Is the GATT/WTO's Article XXIV Bad?: Online Appendix Not For Publication

Monika Mrázová*
University of Surrey; and CEP, LSE

David Vines†
University of Oxford; Australian National University; and CEPR

Ben Zissimos‡
University of Exeter

August 1, 2012

Abstract

This online Appendix includes various proofs and details that were left out of our manuscript "Is the GATT/WTO's Article XXIV Bad?" due to space constraints.

For all the derivations, unless otherwise stated, we assume that the parameters of the model lie within the relevant ranges for the analysis, namely $\gamma \in [0,1]$, $N > 1$ and $k \in [1, N]$.

A Properties of k^* and k^{**}

To get an idea for which range of parameters the Article XXIV constraint binds, namely where $k^{**} > 1$, it is useful to study the variations of k^* and k^{**} with γ and N. It is immediate to see that both k^* and k^{**} are increasing functions of N. The following shows that they are decreasing functions of γ:

$$\frac{\partial k^*(N,\gamma)}{\partial \gamma} = \frac{\Phi_{k^*}(N,\gamma)}{4\gamma^2\sqrt{\Gamma(0)\left[\Gamma(0)+1\right]\Gamma(N)}}$$

with $\Phi_{k^*}(N,\gamma) \equiv 4\sqrt{\Gamma(0)\left[\Gamma(0)+1\right]\Gamma(N)} + \gamma^3(N-1) + 2\gamma(8-3N) - 24$. The denominator being strictly positive, the derivative of k^* is of the same sign as its numerator Φ_{k^*}. Furthermore, $\frac{\partial \Phi_{k^*}}{\partial N}(N,\gamma) = \gamma\left(-6+\gamma^2+2\sqrt{\frac{\Gamma(0)[\Gamma(0)+1]}{\Gamma(N)}}\right)$ which is strictly negative for $0 \le \gamma \le 1$ and

*Corresponding author: School of Economics, University of Surrey, Guildford, Surrey, GU2 7XH, United Kingdom; e-mail: m.mrazova@surrey.ac.uk; URL: http://www.monikamrazova.com

†E-mail: david.vines@economics.ox.ac.uk

‡E-mail: b.zissimos@exeter.ac.uk

$N \geq 1$. So Φ_{k^*} is a decreasing function of N and $\Phi_{k^*}(1,\gamma) = 4\sqrt{2(2-\gamma)(3-\gamma)}+10\gamma-24 < 4\sqrt{12}+10-24 < 0$ and so Φ_{k^*} is always negative and k^* is a monotonically decreasing function of γ. When $\gamma = 0$, k^* is infinite, when $\gamma = 1$, $k^* = \frac{\sqrt{2(N+1)}-1}{2} > 0$. So $k^* > 1$ for any γ and $N \geq 4$. Furthermore

$$\frac{\partial k^{**}(N,\gamma)}{\partial \gamma} = \frac{\Phi_{k^{**}}(N,\gamma)}{2\gamma^2\Gamma(2)^2}$$

with $\Phi_{k^{**}}(N,\gamma) \equiv -16 - 16\gamma - 16(N-2)\gamma^2 + 4(N-1)\gamma^3 + (N-1)\gamma^4$. The denominator being strictly positive, the derivative of k^{**} is of the same sign as its numerator $\Phi_{k^{**}}$. Note that

$$\begin{aligned}\Phi_{k^{**}}(N,\gamma) \leq & -16 - 16\gamma - 16(N-2)\gamma^2 + 4(N-1)\gamma^2 + (N-1)\gamma^2 \\ \leq & -\left[16 + 16\gamma + (11N-27)\gamma^2\right] \leq -16\left[1+\gamma-\gamma^2\right] < 0\end{aligned}$$

and so k^{**} is a decreasing function of γ. When $\gamma = 0$, k^{**} is infinite, when $\gamma = 1$, $k^{**} = \frac{2N-1}{6} > 0$. So $k^{**} > 1$ for any γ and $N \geq 4$, $k^{**} > 2$ for any γ and $N \geq 7$.

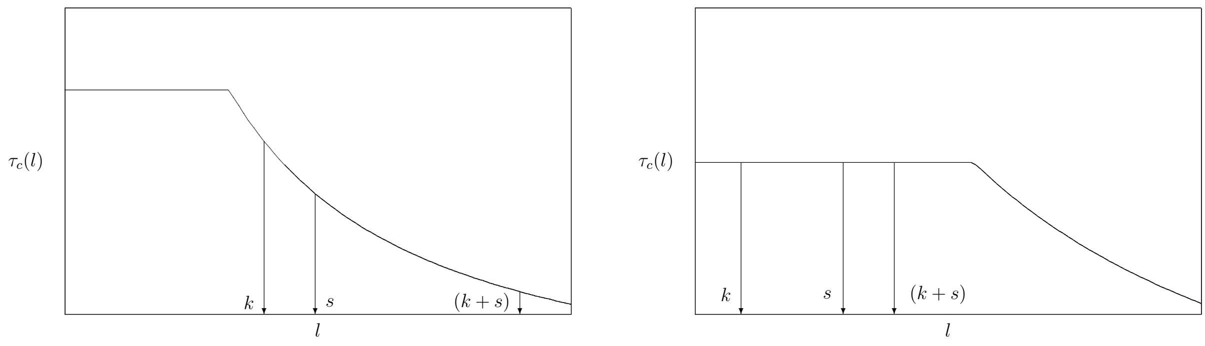
B Proofs from Section 3

Proof of Lemma 3. We want to prove that formation or expansion of CUs under the Article XXIV constraint increases the aggregate welfare of member countries. To do so, we suppose that CUs of size k, l, m, $\ldots$, and size r merge and we show that the aggregate welfare of the countries involved in the merger increases. Without loss of generality, we consider the merger of a size$-k$ CU and a size$-s$ CU where $s = l + m + \ldots + r$.

The proof consists of three steps: First, to prove the case where the Article XXIV constraint does not bind on the CUs involved in the merger, we invoke the proof of Yi's (1996) Proposition 3; the case shown in Figure 1(a). Note that it is valid to consider a group of CUs for whom the Article XXIV constraint does not bind even though it might bind on other CUs not involved in the merger. Second, we prove the proposition for CUs on which the Article XXIV constraint is binding as shown in Figure 1(b). Finally, we will show how the first two sub-cases generalize for any CU merger.

Step 1: merger of CUs that are not constrained by Article XXIV - see Yi (1996), Appendix B, pages 172–175

Step 2: merger of CUs that are constrained by Article XXIV
Here we consider the merger of size$-k$ and size$-s$ CUs such that the Article XXIV constraint is binding for both the two individual CUs and for the resulting size$-(k+s)$ CU. The proof proceeds similarly to Yi's case of unconstrained CUs. The goal of the proof is to show the following claim: Suppose that country i has free trade with $k-1$ countries and levies equal tariffs $\tau_c(k) = \tau(1)$ on $N-k$ countries. If country i abolishes tariffs on s countries,



(a) Article XXIV neither binding before nor after

(b) Article XXIV both binding before and after

Figure 1: Two different merger situations: external tariffs of original and resulting CUs.

$s \leq N-k$, and levies $\tau_c(k+s) = \tau(1)$ on the remaining $N-k-s$ countries, then the aggregate welfare of $k+s$ countries (which consist of country i, $k-1$ countries which pay no tariffs, and s countries whose tariffs are eliminated) improves.

Without loss of generality, take country 1 and suppose that it levies no tariffs on countries $2, \ldots, k$, and $\tau_c(k) = \tau(1)$ on countries $k+1, \ldots, N$. We are interested in the following comparative statics exercise: what is the effect on the aggregate welfare of countries $1, \ldots, k+s$ of abolishing tariffs on countries $k+1, \ldots, k+s$ and keeping tariffs on countries $k+s+1, \ldots, N$ at $\tau_c(k+s) = \tau(1) = \tau_c(k)$?

Using the same notation as Yi (1996), consider a tariff vector

$$\mathbf{t} \equiv (0, \ldots, 0, \tau, \ldots, \tau, \tau', \ldots, \tau') \tag{B.1}$$

where τ appears from the $(k+1)$th column to the $(k+s)$th column and τ' from the $(k+s+1)$th column to the last column. Consider the following two tariff vectors: $\mathbf{t_c}(k+s) \equiv [0, \ldots, 0, \tau_c(k+s), \ldots, \tau_c(k+s)]$ with 0 in the first $(k+s)$ columns and $\mathbf{t_c}(k) \equiv [0, \ldots, 0, \tau_c(k), \ldots, \tau_c(k)]$ with 0 from the first to the kth column (where $\tau_c(k+s) = \tau_c(k) = \tau(1)$). We can move from $\mathbf{t_c}(k+s)$ to $\mathbf{t_c}(k)$ by integrating from 0 to $\tau_c(k)$ the infinitesimal changes from the tariff vector defined by (B.1) $d\mathbf{t} \equiv (0, \ldots, 0, d\tau, \ldots, d\tau, d\tau', \ldots, d\tau')$ with $d\tau' = 0$: $\mathbf{t_c}(k) = \mathbf{t_c}(k+s) + \int_0^{\tau_c(k)} d\mathbf{t}$.

To prove our claim, similarly as Yi (1996), we show that $d(\sum_{j=1}^{k+s} W^j)/d\mathbf{t} < 0$ for all $\mathbf{t}$ along such a path of integration. To do so, we first show that $d(\sum_{j=1}^{k+s} W^j)/d\mathbf{t} < 0$ for $\mathbf{t_c}(k+s) = [0, \ldots, 0, \tau_c(k+s), \ldots, \tau_c(k+s)]$. And second, we show that $d^2(\sum_{j=1}^{k+s} W^j)/d\mathbf{t}^2 < 0$.

Step 2a: Since changes in country 1's tariffs do not affect sales in other countries, $d(\sum_{j=1}^{k+s} W^j)/d\mathbf{t} =$

$d(\hat{W}^1 + \sum_{j=2}^{k+s} \pi^{1j})/d\mathbf{t}$, where $\hat{W}^1$ is country 1's welfare net of its export profits. Since $\hat{W}^1 + \sum_{j=2}^{N} \pi^{1j} = v(\mathbf{q}_1) - cQ_1$, which is the net total benefit from consumption of $\mathbf{q}_1$, $\hat{W}^1 + \sum_{j=2}^{k+s} \pi^{1j} = v(\mathbf{q}_1) - cQ_1 - \sum_{j=k+s+1}^{N} \pi^{1j}$. To save on notation, we can drop superscript 1.

The total tariff T at the tariff vector $\mathbf{t}$ is $T = \sum_{j=1}^{N} \tau_j = s\tau + (N-k-s)\tau'$ and $dT = sd\tau + (N-k-s)d\tau' = sd\tau$. From the first-order-condition of firms' profit maximization, $p_j - c = q_j + \tau_j$. Then $\sum_{j=1}^{N} [p_j - c] = Q + T$. At $\mathbf{t}$, $q_1 = \ldots = q_k$, $q_{k+1} = \ldots = q_{k+s}$ and $q_{k+s+1} = \ldots = q_N$. From (2), $dq_j = \frac{\gamma dT - \Gamma(N) d\tau_j}{\Gamma(0)\Gamma(N)}$. Thus,

$$\frac{dq_1}{d\mathbf{t}} = \frac{\gamma s}{\Gamma(0)\Gamma(N)}, \ \frac{dq_{k+1}}{d\mathbf{t}} = \frac{\gamma s - \Gamma(N)}{\Gamma(0)\Gamma(N)} \text{ and } \frac{dq_N}{d\mathbf{t}} = \frac{\gamma s}{\Gamma(0)\Gamma(N)}$$

Using these results,

$$\begin{aligned}\frac{d}{d\mathbf{t}}\left(\hat{W} + \sum_{j=2}^{k+s} \pi^j\right) &= \frac{d}{d\mathbf{t}}\left[v(\mathbf{q}) - cQ\right] - \frac{d}{d\mathbf{t}} \sum_{j=k+s+1}^{N} \pi^j = \sum_{j=1}^{N} [p_j - c]\frac{dq_j}{d\mathbf{t}} - \sum_{j=k+s+1}^{N} 2q_j \frac{dq_j}{d\mathbf{t}} \\ &= \frac{s\Omega}{\Gamma(0)\Gamma(N)}\end{aligned}$$

where $\Omega \equiv \gamma(Q+T) - \Gamma(N)\left[q_{k+1} + \tau\right] - 2(N-k-s)\gamma q_N$. At $\mathbf{t_c}(k+s)$, $\Omega = -\Gamma(0)q_1 - (N-k-s)\gamma\left[q_1 + q_N - \tau_c(k+s)\right]$, and

$$\begin{aligned}q_1 + q_N - \tau_c(k+s) &= \frac{1}{\Gamma(0)\Gamma(N)}\{2\Gamma(0) - \left[\Gamma(N)\left[\Gamma(0)+1\right] - 2(N-k-s)\gamma\right]\tau_c(k+s)\} \\ &= \frac{1}{D(1)\Gamma(N)}\left[\underbrace{12 - 12\gamma + 9\gamma^2 - \gamma^3 + \gamma(6 - 5\gamma + \gamma^2)N}_{\geq 0} - 2\gamma(2+\gamma)(k+s)\right]\end{aligned}$$

The expression in the square bracket is a linear, decreasing function of $k+s$. As we are assuming that Article XXIV binds, we must have $k+s \leq k^{**}$. Hence

$$\begin{aligned}q_1 + q_N - \tau_c(k+s) &\geq \frac{1}{D(1)\Gamma(N)}\left[12 - 12\gamma + 9\gamma^2 - \gamma^3 + \gamma(6 - 5\gamma + \gamma^2)N - 2\gamma(2+\gamma)k^{**}\right] \\ &= \frac{(2+\gamma)^2}{D(1)\Gamma(N)} > 0\end{aligned}$$

Step 2b: Again following Yi (1996), we have

$$\frac{d^2}{d\mathbf{t}^2}\left(\hat{W}+\sum_{j=2}^{k+s}\pi^j\right) = -\frac{s}{\Gamma(0)^2\Gamma(N)^2}\left\{(1-\gamma)\Gamma(N)\Gamma(N-s)+s\gamma\left[\Gamma(0)+2(N-k-s)\gamma\right]\right\}<0$$

Step 3: generalization

So far we have proved the proposition in two different situations: first, when the Article XXIV constraint is not binding (neither for the initial CUs nor for the after-merger CU); second when the Article XXIV constraint is binding in both cases. We now have to show that the result still holds when the Article XXIV constraint is binding for (at least one of) the initial CUs but not for after-merger CU. (It is easy to deduce from Figures 3a and b that the converse cannot occur). This step is an easy comparative statics exercise. Consider the tariffs of the CUs not involved in the merger as given (could be constrained or unconstrained). The subscript u denotes an unconstrained CU and the subscript c denotes a constrained CU. By Yi's proof, we have $(k+s)W_u(k+s) \geq kW_u(k)+sW_u(s)$. Now given the tariffs of the CUs not involved in this merger, the CUs of size-k and size-s involved in this merger are better off when unconstrained compared to the constrained situation $kW_u(k)+sW_u(s) \geq kW_c(k)+sW_c(s)$. So, we have $(k+s)W_u(k+s) \geq kW_c(k)+sW_c(s)$. $\square$

C Proofs from Section 4

Proof of Proposition 3. The goal here is to determine the impact on world welfare of an expansion of a CU. Assume that $C = \{k_1, k_2, \ldots, k_m\}$ is the CU structure. Without loss of generality, assume that the first CU (of size k_1) is expanding by accepting members from the last CU (of size k_m). We want to see the impact of an increase in the asymmetry between these two CUs, so we assume $k_1 \geq k_m$. The large CU is imposing a CET τ_1, the small CU is imposing τ_m. The actual sub-structure $\{k_2, \ldots, k_{m-1}\}$, which stays constant, will be irrelevant for the changes in world welfare and the only thing that will matter will be the sum of the sizes of the other unions so let's define $\tilde{k} \equiv k_2 + \ldots + k_{m-1}$. We can then re-express the size of the last CU $k_m = N - \tilde{k} - k_1$. We want to determine the sign of $\frac{dW_W(k_1,k_2,\ldots,N-\tilde{k}-k_1)}{dk_1}$. The total derivative can be decomposed as follows

$$\frac{dW_W}{dk_1} = \left.\frac{\partial W_W}{\partial k_1}\right|_{(\tau_1,\tau_m)} + \underbrace{\frac{\partial W_W}{\partial \tau_1}}_{\leq 0}\underbrace{\frac{\partial \tau_1}{\partial k_1}}_{\leq 0} + \frac{\partial W_W}{\partial \tau_m}\underbrace{\frac{\partial \tau_m}{\partial k_1}}_{=0} \tag{C.1}$$

We assume that Article XXIV binds at least on the small union involved in the transformation, and so we have $\tau_m = \tau_c(N-\tilde{k}-k_1) = \tau(1)$ and hence $\frac{\partial \tau_m}{\partial k_1} = 0$. Therefore the last term in (C.1) is zero. (Note that it is irrelevant whether Article XXIV binds or not on the CUs not involved in the change considered.) Furthermore, with Article XXIV in place, we know that $\tau_1 = \tau_c(k_1)$ is a non-increasing function of k_1 and so we have $\frac{\partial \tau_1}{\partial k_1} \leq 0$. From the proof of Proposition 2 above we also have $\frac{\partial W_W}{\partial \tau_1} \leq 0$. The remainder of this proof shows that $\left.\frac{\partial W_W}{\partial k_1}\right|_{(\tau_1,\tau_m)} \geq 0$ when $k_1 + k_m \geq \frac{2}{3}N$.

From (9) we have

$$W_W = k_1 NS(k_1) + (N - \tilde{k} - k_1)NS(N - \tilde{k} - k_1) + \sum_{i=2}^{m-1} k_i NS(k_i) \tag{C.2}$$

Using (6), (7) and (8) we have

$$\begin{aligned} NS(k_i) = \frac{1}{2\Gamma(0)^2\Gamma(N)^2} \{ & N\Gamma(0)^2 \left[\Gamma(N) + 1\right] - 2\Gamma(0)^2(N - k_i)\tau_i \\ & + (N - k_i)\left[D(k_i) - 2\Gamma(k_i)^2\right]\tau_i^2 \} \end{aligned} \tag{C.3}$$

Substituting (C.3) into (C.2) yields

$$\begin{aligned} W_W = \frac{1}{2\Gamma(0)^2\Gamma(N)^2} \{ & \; (N - \tilde{k})N\Gamma(0)^2 \left[\Gamma(N) + 1\right] \\ & -2\Gamma(0)^2 \left[k_1(N - k_1)\tau_1 + (N - \tilde{k} - k_1)(\tilde{k} + k_1)\tau_m\right] \\ & +k_1(N - k_1)\left[D(k_1) - 2\Gamma(k_1)^2\right]\tau_1^2 \\ & +(N - \tilde{k} - k_1)(\tilde{k} + k_1)\left[D(N - \tilde{k} - k_1) - 2\Gamma(N - \tilde{k} - k_1)^2\right]\tau_m^2 \Big\} \\ & + \sum_{i=2}^{m-1} k_i NS(k_i) \end{aligned}$$

Differentiating with respect to k_1 yields

$$\begin{aligned} \left.\frac{\partial W_W}{\partial k_1}\right|_{(\tau_1,\tau_m)} = \frac{1}{2\Gamma(0)^2\Gamma(N)^2} \{ & 2\Gamma(0)^2 \left[(2k_1 - N)\tau_1 + \tilde{k}\tau_m + (2k_1 + \tilde{k} - N)\tau_m\right] \\ & +\alpha(k_1, N, \gamma)\tau_1^2 + \beta(\tilde{k} + k_1, N, \gamma)\tau_m^2 \Big\} \end{aligned} \tag{C.4}$$

where

$$\begin{aligned} \alpha(k, N, \gamma) \equiv & -3\gamma^2\left[\Gamma(N) - N\right]k^2 \\ & + 2\left[-\gamma^2(1-\gamma)N^2 + 2\gamma\Gamma(0)N + \Gamma(0)^2(1-\gamma)\right]k \\ & - \Gamma(0)^2(1 - \gamma + \gamma N)N \\ \beta(k, N, \gamma) \equiv & \; 3\gamma^2\left[\Gamma(N) - N\right]k^2 \\ & + 2\left[2\gamma^2(1-\gamma)N^2 + \gamma\Gamma(0)(2 - 3\gamma)N + \Gamma(0)^2(1-\gamma)\right]k \\ & - (1-\gamma)\Gamma(N)^2 N \end{aligned}$$

Recall from the study of the Article XXIV constraint that Article XXIV can either bind on both the small and the large union ($\tau_1 = \tau_m = \tau(1)$) or it can bind on the small union only ($\tau_m = \tau(1)$ and $\tau_1 = \tau(k_1) \leq \tau(1)$). Article XXIV can never bind on the large union and not bind on the small union. To sign (C.4) we thus need to distinguish two cases:

1) Article XXIV binds on both the small and the large CUs ($\tau_1 = \tau_m = \tau(1)$): When both unions are constrained, equation (C.4) simplifies to

$$\left.\frac{\partial W_W}{\partial k_1}\right|_{(\tau_1,\tau_m)} = \frac{(k_1-k_m)\tau(1)}{2\Gamma(0)^2\Gamma(N)^2}\Big\{4\Gamma(0)^2 + \left[\Gamma(0)^2(1-\gamma+\gamma N) + (1-\gamma)\Gamma(N)^2\right]\tau(1) \\ - \tilde{k}3\gamma^2\left[N-\Gamma(N)\right]\tau(1)\Big\} \tag{C.5}$$

The first line of (C.5) is positive. The second line can be either positive or negative depending on the parameters γ and N:

$$N - \Gamma(N) \leq 0 \Leftrightarrow \gamma \geq \bar{\gamma}(N) \equiv \frac{N-2}{N-1} \tag{C.6}$$

Hence $\left.\frac{\partial W_W}{\partial k_1}\right|_{(\tau_1,\tau_m)}$ is unambiguously positive for $\gamma \geq \bar{\gamma}(N)$. For $\gamma = 0$, the expression of the derivative (C.5) further simplifies and it is also unambiguously positive: $\left.\frac{\partial W_W}{\partial k_1}\right|_{(\tau_1,\tau_m)} = \frac{(k_1-k_m)\tau(1)}{4}\left[2+\tau(1)\right] \geq 0$.

For $0 < \gamma < \bar{\gamma}(N)$, the second line of (C.5) is negative. The whole expression $\left.\frac{\partial W_W}{\partial k_1}\right|_{(\tau_1,\tau_m)}$ is positive provided that $\tilde{k}$ is sufficiently small. In other words, the two CUs involved in the change considered have to represent a sufficient proportion of countries in the world: we have to have $k_1 + k_m = N - \tilde{k} \geq \bar{k}_c \equiv \max(0, \hat{k}_c)$ where

$$\begin{aligned}\hat{k}_c &\equiv N - \frac{4\Gamma(0)^2 + \left[\Gamma(0)^2(1-\gamma+\gamma N) + (1-\gamma)\Gamma(N)^2\right]\tau(1)}{3\gamma^2\left[N-\Gamma(N)\right]\tau(1)} \\ &= \frac{2\left\{(1-\gamma)\gamma^2\tau(1)N^2 - 2\Gamma(0)\gamma\tau(1)N - \Gamma(0)^2\left[2+(1-\gamma)\tau(1)\right]\right\}}{3\left[(1-\gamma)\gamma^2\tau(1)N - \Gamma(0)\gamma^2\tau(1)\right]}\end{aligned}$$

What can we say of $\bar{k}_c$? The denominator of $\hat{k}_c$ is strictly positive on the range considered ($0 < \gamma < \bar{\gamma}(N)$). When the numerator of $\hat{k}_c$ is negative, $\bar{k}_c = 0$, and the derivative of welfare is positive for any value of $\tilde{k}$ up to N (the minimum size of the two CUs is zero). When the numerator of $\hat{k}_c$ is strictly positive, $\bar{k}_c > 0$. Furthermore, as $-\Gamma(0)\gamma^2\tau(1) \geq -2\Gamma(0)\gamma\tau(1)$, we have

$$\hat{k}_c \leq \frac{2\left[(1-\gamma)\gamma^2\tau(1)N - 2\Gamma(0)\gamma\tau(1)\right]N}{3\left[(1-\gamma)\gamma^2\tau(1)N - 2\Gamma(0)\gamma\tau(1)\right]} = \frac{2}{3}N$$

Hence $0 \leq \bar{k}_c \leq \frac{2}{3}N$.

2) Article XXIV binds on the small union only ($\tau_1 = \tau(k_1) \leq \tau(1) = \tau_m$): With $\tau_m \geq \tau_1$, we have that

$$\begin{aligned}(2k_1 - N)\tau_1 + \tilde{k}\tau_m + (2k_1 + \tilde{k} - N)\tau_m &\geq (2k_1 - N)\tau_1 + \tilde{k}\tau_1 + (2k_1 + \tilde{k} - N)\tau_m \\ &= (k_1 - k_m)(\tau_1 + \tau_m) \geq 0\end{aligned}$$

And so the first line of (C.4) is unambiguously positive. We now need to sign the expressions α and β. Before we proceed to do so, it is useful to recall certain conditions that have to be satisfied when Article XXIV binds on the small union, but not on the large. First, we must necessarily have $\gamma > \gamma_{N\infty} = \frac{7-\sqrt{41}}{2}$. Furthermore, for the large union not to be constrained, we must have $k_1 \geq k^{**} \Rightarrow \tilde{k} + k_1 \geq k^{**}$. On the other hand, for the small union to be constrained, we must have $k_m = N - \tilde{k} - k_1 \leq k^{**} \Leftrightarrow \tilde{k} + k_1 \geq N - k^{**}$. Which of these two conditions binds depends on γ. From the study of k^{**} we know that:

- For $\gamma \leq \gamma_{\frac{N}{2}} = 3 - \sqrt{5}$, $k^{**} \geq N - k^{**}$, and so we have to have $\tilde{k} + k_1 \geq k^{**} \geq N - k^{**}$.
- For $\gamma > \gamma_{\frac{N}{2}} = 3 - \sqrt{5}$, $k^{**} < N - k^{**}$, and so we have to have $\tilde{k} + k_1 \geq N - k^{**} > k^{**}$.

We now proceed to sign α and β:

2.1) We show that $\beta(k, N, \gamma) \geq 0$ for any $N \geq 6$, $\gamma \in [\gamma_{N\infty}, 1]$ and $k \geq \max(k^{**}, N - k^{**})$: $\beta(k, N, \gamma)$ is a second degree polynomial in k. To sign it we differentiate successively with respect to k.

$$\frac{\partial^2 \beta(k, N, \gamma)}{\partial k^2} = 6\gamma^2 \left[\Gamma(N) - N\right]$$

From (C.6) we know that this second derivative can be either positive or negative depending on the parameters N and γ. We thus need to distinguish two sub-cases:

2.1.1) For $\gamma > \bar{\gamma}(N)$: $\frac{\partial^2 \beta(k,N,\gamma)}{\partial k^2} > 0$ and so $\frac{\partial \beta(k,N,\gamma)}{\partial k}$ is an increasing function of k. To sign this first derivative, we evaluate it at the lower bound of our interval of interest. Assuming $N \geq 6$, we have $\bar{\gamma}(N) > \gamma_{\frac{N}{2}}$. We are thus interested in $k = \tilde{k} + k_1 \geq N - k^{**}$.

$$\begin{aligned}\frac{\partial}{\partial k}\beta(N - k^{**}, N, \gamma) = \frac{1}{2+\gamma}\Bigg[&\underbrace{(1-\gamma)^2\gamma^2(14 - 3\gamma)}_{\geq 0 \text{ for } \gamma \in [0,1]} N^2 + \underbrace{2\gamma\Gamma(0)(10 - 22\gamma + 18\gamma^2 - 3\gamma^3)}_{\geq 0 \text{ for } \gamma \in [0,1]} N \\ &+ \underbrace{\Gamma(0)^2(4 - 14\gamma + 16\gamma^2 - 3\gamma^3)}_{\geq 0 \text{ for } \gamma \in [0,1]}\Bigg] \geq 0 \qquad \text{(C.7)}\end{aligned}$$

and so $\frac{\partial \beta(k,N,\gamma)}{\partial k}$ is positive for any $k \geq N - k^{**}$.[1] Hence β is an increasing function of k for

[1] This proof and many of the following proofs require us to sign various polynomial functions of γ like

$k \geq N - k^{**}$. Again, to sign β, we evaluate it at the lower bound $k = N - k^{**}$:

$$\beta(N - k^{**}, N, \gamma) = \frac{\Gamma(0)}{4\gamma(2+\gamma)^2} g_\beta(N, \gamma)$$

with

$$\begin{aligned} g_\beta(N,\gamma) \equiv\ & +\gamma^3(1-\gamma)(3-\gamma)(-10+19\gamma-3\gamma^2)N^3 \\ & -\gamma^2(1-\gamma)(176-396\gamma+328\gamma^2-97\gamma^3+9\gamma^4)N^2 \\ & +\gamma\Gamma(0)(4-6\gamma+\gamma^2)(-36+62\gamma-47\gamma^2+9\gamma^3)N \\ & +\Gamma(0)^2(4-6\gamma+\gamma^2)(-8+16\gamma-14\gamma^2+3\gamma^3) \end{aligned}$$

$g_\beta(N,\gamma)$ is a third degree polynomial in N. To determine its sign we differentiate successively with respect to N. $\frac{\partial^3 g_\beta(N,\gamma)}{\partial N^3} = 6\gamma^3(1-\gamma)(3-\gamma)(-10+19\gamma-3\gamma^2) \geq 0$ for $\gamma \geq \gamma_{\frac{N}{2}}$ so $\frac{\partial^2 g_\beta(N,\gamma)}{\partial N^2}$ is an increasing function of N. As $\frac{\partial^2}{\partial N^2} g_\beta(6,\gamma) = 2\gamma^2(1-\gamma)(-176-144\gamma+878\gamma^2-407\gamma^3+45\gamma^4) \geq 0$ for $\gamma \geq \gamma_{\frac{N}{2}}$, $\frac{\partial^2 g_\beta(N,\gamma)}{\partial N^2}$ is positive for any $N \geq 6$ and $\frac{\partial g_\beta(N,\gamma)}{\partial N}$ is an increasing function of N for $N \geq 6$. Furthermore, $\frac{\partial g_\beta(6,\gamma)}{\partial N} = \gamma(-288-1040\gamma+1968\gamma^2+3144\gamma^3-5742\gamma^4+2195\gamma^5-225\gamma^6) \geq 0$ for $\gamma \geq \gamma_{\frac{N}{2}}$ and so the first derivative is positive for any $N \geq 6$ and $g_\beta(N,\gamma)$ is an increasing function of N. Finally, $g_\beta(6,\gamma) = -128-1152\gamma-1024\gamma^2+5376\gamma^3+2288\gamma^4-8460\gamma^5+3550\gamma^6-375\gamma^7 \geq 0$ for $\gamma \geq \gamma_{\frac{N}{2}}$ and so $g_\beta(N,\gamma)$ is positive for any $N \geq 6$ and $\gamma \geq \gamma_{\frac{N}{2}}$. Thus we have that $\beta(N-k^{**},N,\gamma) \geq 0$ and so $\beta(k,N,\gamma)$ is positive for any $k \geq N-k^{**}$, $N \geq 6$ and $\gamma > \bar{\gamma}(N)$.

2.1.2) For $\boldsymbol{\gamma \leq \bar{\gamma}(N)}$: $\frac{\partial^2 \beta(k,N,\gamma)}{\partial k^2} \leq 0$ and so $\frac{\partial \beta(k,N,\gamma)}{\partial k}$ is a decreasing function of k. We need to distinguish two further sub-cases:

2.1.2.1) $\boldsymbol{\gamma_{\frac{N}{2}} \leq \gamma \leq \bar{\gamma}(N)}$: We know from (C.7) that $\frac{\partial}{\partial k}\beta(N-k^{**},N,\gamma)$ is positive. On the other hand $\frac{\partial}{\partial k}\beta(N,N,\gamma)$ can be either positive or negative for $\gamma \leq \bar{\gamma}(N)$. Therefore, $\beta(k,N,\gamma)$ is either an increasing function of k or it is initially an increasing and then decreasing function of k. In either of these two cases, if we show that $\beta(k,N,\gamma)$ is positive on the bounds of the interval $[N-k^{**},N]$, we will know that it is positive for any k in this interval. From the above, we know already that at the lower bound $\beta(N-k^{**},N,\gamma) \geq 0$ for $\gamma \geq \gamma_{\frac{N}{2}}$. Furthermore, we have at the upper bound $\beta(N,N,\gamma) = \Gamma(0)^2 N(1-\gamma+\gamma N) \geq 0$. And so $\beta(k,N,\gamma)$ is positive for any $k \geq N-k^{**}$, $N \geq 6$ and $\gamma \geq \gamma_{\frac{N}{2}}$.

2.1.2.2) $\boldsymbol{\gamma < \gamma_{\frac{N}{2}} \leq \bar{\gamma}(N)}$: When $\gamma_{\frac{N}{2}} \leq \bar{\gamma}(N)$, we have $k^{**} \geq N-k^{**}$ and we are therefore interested in signing β for $k \in [k^{**},N]$. From (C.7) we know that $\frac{\partial}{\partial k}\beta(N-k^{**},N,\gamma)$ is positive and as we already mentioned $\frac{\partial}{\partial k}\beta(N,N,\gamma)$ can be either positive or negative for $\gamma \leq \bar{\gamma}(N)$. As $\frac{\partial \beta(k,N,\gamma)}{\partial k}$ is a decreasing function of k, we have $\frac{\partial}{\partial k}\beta(N-k^{**},N,\gamma) \geq$

$4-14\gamma+16\gamma^2-3\gamma^3$ above. All these functions are continuous and differentiable. They are function of a single variable γ taking values in a bounded interval $[0,1]$ or parts of this interval. The sign of these functions can be determined by successive differentiation with respect to γ. For the sake of space, we will not present these detailed differentiations.

$\frac{\partial}{\partial k}\beta(k^{**}, N, \gamma)$. If $\frac{\partial}{\partial k}\beta(k^{**}, N, \gamma) \leq 0$ then we necessarily have $\frac{\partial}{\partial k}\beta(N, N, \gamma) \leq 0$ and the first derivative is negative on the whole range of interest. In this case, $\beta(k, N, \gamma)$ is a decreasing function of k on $[k^{**}, N]$. If $\frac{\partial}{\partial k}\beta(k^{**}, N, \gamma) \geq 0$, we can still have $\frac{\partial}{\partial k}\beta(N, N, \gamma)$ of either sign in which case $\beta(k, N, \gamma)$ is either monotonically increasing or initially increasing and then decreasing in $[k^{**}, N]$. In any of these three cases, if we show that $\beta(k, N, \gamma)$ is positive on the bounds of the interval $[k^{**}, N]$, we will know that it is positive for any k in this interval. We already know that $\beta(N, N, \gamma) \geq 0$. We now need to sign $\beta(k^{**}, N, \gamma)$.

$$\beta(k^{**}, N, \gamma) = \frac{1}{4\gamma(2+\gamma)^2} f_\beta(N, \gamma)$$

with

$$\begin{aligned} f_\beta(N, \gamma) \equiv & -\gamma^3(1-\gamma)^2(14-3\gamma)(2-7\gamma+\gamma^2)N^3 \\ & +\gamma^2\Gamma(0)(-16+284\gamma-676\gamma^2+509\gamma^3-122\gamma^4+9\gamma^5)N^2 \\ & +\gamma(1-\gamma)\Gamma(0)^2(4-6\gamma+\gamma^2)(12+58\gamma-9\gamma^2)N \\ & +\Gamma(0)^3(4-6\gamma+\gamma^2)(8+8\gamma-22\gamma^2+3\gamma^3) \end{aligned}$$

$f_\beta(N, \gamma)$ is a third degree polynomial on N. To determine its sign we differentiate successively with respect to N. $\frac{\partial^3 f_\beta(N,\gamma)}{\partial N^3} = -6\gamma^3(1-\gamma)^2(14-3\gamma)(2-7\gamma+\gamma^2) \geq 0$ for $\gamma \geq \gamma_{N\infty}$ so $\frac{\partial^2 f_\beta(N,\gamma)}{\partial N^2}$ is an increasing function of N. As $\frac{\partial^2}{\partial N^2} f_\beta(6, \gamma) = 2\gamma^2(-32+80\gamma+1244\gamma^2-3184\gamma^3+2433\gamma^4-598\gamma^5+45\gamma^6) \geq 0$ for $\gamma \in [\gamma_{N\infty}, \gamma_{\frac{N}{2}}]$, $\frac{\partial^2 f_\beta(N,\gamma)}{\partial N^2}$ is positive for any $N \geq 6$ and $\frac{\partial f_\beta(N,\gamma)}{\partial N}$ is an increasing function of N for $N \geq 6$. Furthermore, $\frac{\partial f_\beta(6,\gamma)}{\partial N} = \gamma(192-128\gamma+1456\gamma^2+1824\gamma^3-11892\gamma^4+11084\gamma^5-2905\gamma^6+225\gamma^7) \geq 0$ for $\gamma \in [\gamma_{N\infty}, \gamma_{\frac{N}{2}}]$ and so the first derivative is positive for any $N \geq 6$ and $f_\beta(N, \gamma)$ is an increasing function of N. Finally, $f_\beta(6, \gamma) = 256+640\gamma-256\gamma^2+2432\gamma^3-2176\gamma^4-13720\gamma^5+16720\gamma^6-4700\gamma^7+375\gamma^8 \geq 0$ for $\gamma \in [\gamma_{N\infty}, \gamma_{\frac{N}{2}}]$ and so $f_\beta(N, \gamma)$ is positive for any $N \geq 6$ and $\gamma \in [\gamma_{N\infty}, \gamma_{\frac{N}{2}}]$. Thus we have that $\beta(k^{**}, N, \gamma) \geq 0$ and so $\beta(k, N, \gamma)$ is positive for any $k \geq k^{**}$, $N \geq 6$ and $\gamma \in [\gamma_{N\infty}, \gamma_{\frac{N}{2}}]$.

Hence we have shown that $\beta(k, N, \gamma)$ is positive for $N \geq 6$, $\gamma \in [\gamma_{N\infty}, 1]$ and $k \geq \max(k^{**}, N-k^{**})$ which covers the entire relevant range where the Article XXIV constraint might bind on the small union, but not on the large union.

2.2) $\boldsymbol{\alpha(k, N, \gamma)}$: Similar derivations as for $\beta(k, N, \gamma)$ show that $\alpha(k, N, \gamma)$ can be either positive or negative depending on the parameters. We can therefore distinguish two cases:

2.2.1) When $\boldsymbol{\alpha(k, N, \gamma) \geq 0}$: We can immediately conclude that $\left.\frac{\partial W_W}{\partial k_1}\right|_{(\tau_1, \tau_m)}$ is positive.

2.2.1) When $\boldsymbol{\alpha(k, N, \gamma) < 0}$: We can note that

$$\alpha(k_1, N, \gamma)\tau_1^2 + \beta(\tilde{k}+k_1, N, \gamma)\tau_m^2 \geq \left[\alpha(k_1, N, \gamma) + \beta(\tilde{k}+k_1, N, \gamma)\right]\tau_m^2$$

and

$$\left.\frac{\partial W_W}{\partial k_1}\right|_{(\tau_1,\tau_m)} \geq \frac{(k_1-k_m)}{2\Gamma(0)^2\Gamma(N)^2}\Big\{2\Gamma(0)^2(\tau_1+\tau_m)+\left[\Gamma(0)^2(1-\gamma+\gamma N)+(1-\gamma)\Gamma(N)^2\right]\tau_m^2$$
$$-\tilde{k}3\gamma^2\left[N-\Gamma(N)\right]\tau_m^2\Big\} \tag{C.8}$$

As in (C.5) in the constrained case 1), the first line of (C.8) is positive. The second line can be either positive or negative depending on the parameters γ and N. The second line is positive for $\gamma > \bar{\gamma}(N)$ and for $\gamma = 0$, and thus the partial derivative of welfare is positive for these values. For $0 < \gamma < \bar{\gamma}(N)$, the second line of (C.8) is negative. The whole expression $\left.\frac{\partial W_W}{\partial k_1}\right|_{(\tau_1,\tau_m)}$ is again positive provided that $\tilde{k}$ is sufficiently small. We have to have $k_1 + k_m = N - \tilde{k} \geq \bar{k}_u \equiv \max(0, \hat{k}_u)$ where

$$\begin{aligned}\hat{k}_u &\equiv N - \frac{2\Gamma(0)^2(\tau_1+\tau_m)+\left[\Gamma(0)^2(1-\gamma+\gamma N)+(1-\gamma)\Gamma(N)^2\right]\tau_m^2}{3\gamma^2\left[N-\Gamma(N)\right]\tau_m^2}\\ &= \frac{2\left\{(1-\gamma)\gamma^2\tau_m^2N^2-2\Gamma(0)\gamma\tau_m^2N-\Gamma(0)^2\left[\tau_1+\tau_m+(1-\gamma)\tau_m^2\right]\right\}}{3\left[(1-\gamma)\gamma^2\tau_m^2N-\Gamma(0)\gamma^2\tau_m^2\right]}\end{aligned}$$

And we have as in the constrained case 1)

$$\hat{k}_u \leq \frac{2\left[(1-\gamma)\gamma^2\tau_mN-2\Gamma(0)\gamma\tau_m\right]N}{3\left[(1-\gamma)\gamma^2\tau_mN-2\Gamma(0)\gamma\tau_m\right]} = \frac{2}{3}N$$

And so for $k_1 + k_m \geq \frac{2N}{3}$, we have unambiguously in both cases 1) and 2)

$$\frac{dW_W}{dk_1} = \underbrace{\left.\frac{\partial W_W}{\partial k_1}\right|_{(\tau_1,\tau_m)}}_{\geq 0} + \underbrace{\underbrace{\frac{\partial W_W}{\partial \tau_1}}_{\leq 0}\underbrace{\frac{\partial \tau_1}{\partial k_1}}_{\leq 0}}_{\geq 0} + \frac{\partial W_W}{\partial \tau_m}\underbrace{\frac{\partial \tau_m}{\partial k_1}}_{=0} \geq 0$$

$\square$

D Proofs from Section 5

Proof of Lemma 7. 1. Smallest CU: The last union to form must be the smallest since, by Lemma 5, the smallest CU entails the lowest level of welfare for its members. Note that a symmetric CU structure is not an equilibrium outcome. This is also a simple consequence of Lemma 5: if the last two CUs to form are of the same size, then they would be better off by merging. $\square$

2. Second smallest CU: Again, the second smallest CU must be unique, because two symmetric CUs would be better off by merging. Suppose that the second smallest CU has less than k_0 members. Then the members of this union would be better off by admitting (at least) one more member. $\square$

3. Number of equilibrium CUs: The second smallest CU which is the second-to-last to form has at least k_0 members and all the CUs that form before have strictly more members than this CU. Thus there cannot be more than $I(\frac{N}{k_0})$ CUs in equilibrium where $I(\frac{N}{k_0})$ is the next highest integer to $\frac{N}{k_0}$. The goal of this proof is to determine a lower bound for k_0 in order to get an upper bound for the number of equilibrium CUs.

Recall that k_0 is the largest integer such that any size-k CU, $k \leq k_0$, becomes better off by merging with a single-country CU, i.e. $W(k, C) - W(k-1, C') \geq 0$.

When $\gamma = 0$, for any k and N we have $W(k, C) - W(k-1, C') = \frac{7}{72} > 0$ and k_0 is infinite. There will be only one CU of size N in equilibrium when $\gamma = 0$.

When, $\gamma > 0$, treating k as continuous, we want to solve for k_0 such that

$$\begin{aligned} W(k, C) - W(k-1, C') = 0 \\ \Leftrightarrow NS(k) - NS(k-1) - (N-k)q_O(k)^2 + (N-k+1)q_O(k-1)^2 - q_O(1)^2 = 0 \end{aligned} \quad \text{(D.1)}$$

In order to solve for k_0, we need to distinguish three cases:

a) $\boldsymbol{k_0}$ is such that a CU of size $\boldsymbol{k_0}$ is constrained by Article XXIV

If a size-k_0 CU is constrained by Article XXIV then a size-(k_0-1) must also be constrained by Article XXIV. In this case, making use of (6), (7) and (8), together with $\tau_c(k) = \tau_c(k-1) = \tau(1)$, (D.1) becomes

$$\frac{\tau(1)}{2\Gamma(0)^2\Gamma(N)^2}(\omega_2 k^2 + \omega_1 k + \omega_0) = 0 \quad \text{(D.2)}$$

with

$$\begin{aligned} \omega_2 &\equiv 6\gamma^2\tau(1) \geq 0 \\ \omega_1 &\equiv -2\gamma\left\{4\Gamma(0) + \tau(1)\left[(1+\gamma)\gamma N - 8 + 9\gamma - \gamma^2\right]\right\} \leq 0 \\ \omega_0 &\equiv 2\Gamma(0)\Gamma(2N+4) + \tau(1)\left[-(1-\gamma)\gamma^2 N^2 - (1-\gamma)\gamma(4+\gamma)N + \Gamma(0)(2 - 11\gamma + 2\gamma^2)\right] \end{aligned}$$

Substituting $\tau(1)$ into the ω_i and solving (D.2), a second degree polynomial equation in k, yields

$$k_0 = f_{k_0}(N, \gamma) - \sqrt{g_{k_0}(N, \gamma)} \quad \text{(D.3)}$$

where

$$\begin{aligned} f_{k_0}(N, \gamma) &\equiv \frac{\gamma(3-\gamma)(6-\gamma)N + 32 - 22\gamma + 19\gamma^2 - \gamma^3}{6\gamma(2+\gamma)} \\ g_{k_0}(N, \gamma) &\equiv \frac{\theta_2 N^2 + \theta_1 N + \theta_0}{[6\gamma(2+\gamma)]^2} \end{aligned}$$

with

$$\begin{aligned}
\theta_2 &\equiv \gamma^2(156 - 276\gamma + 171\gamma^2 - 24\gamma^3 + \gamma^4) \\
\theta_1 &\equiv 2\gamma(240 - 744\gamma + 632\gamma^2 - 214\gamma^3 + 25\gamma^4 - \gamma^5) \\
\theta_0 &\equiv 352 - 1696\gamma + 2084\gamma^2 - 1068\gamma^3 + 255\gamma^4 - 26\gamma^5 + \gamma^6
\end{aligned}$$

The goal now is to show that k_0 is a decreasing function of γ and so evaluating k_0 at $\gamma = 1$ will give us a lower bound for k_0 for any γ.

Lemma. *For any $N \geq 6$ and $\gamma > 0$, k_0 is a decreasing function of γ.*

Proof. We first look separately at $f_{k_0}(N, \gamma)$ and $g_{k_0}(N, \gamma)$. The first derivative of $f_{k_0}(N, \gamma)$ with respect to γ is negative.

$$\frac{df_{k_0}}{d\gamma}(N, \gamma) = -\frac{\gamma^2(36 - 4\gamma - \gamma^2)N + (2 - \gamma)(32 + 48\gamma - 6\gamma^2 - \gamma^3)}{6\gamma^2(2 + \gamma)^2} < 0$$

So f_{k_0} is a decreasing function of γ. Furthermore,

$$\frac{dg_{k_0}}{d\gamma}(N, \gamma) = \frac{\Phi_{k_0}}{18\gamma^3(2 + \gamma)^3}$$

with

$$\begin{aligned}
\Phi_{k_0} &\equiv \phi_2 N^2 + \phi_1 N + \phi_0 \\
\phi_2 &\equiv -\gamma^3(12 - 12\gamma + \gamma^2)(36 - 4\gamma - \gamma^2) \leq 0 \\
\phi_1 &\equiv -\gamma(2 - \gamma)(240 + 480\gamma - 1136\gamma^2 + 176\gamma^3 + 13\gamma^4 - 2\gamma^5) \lesseqgtr 0 \\
\phi_0 &\equiv -(2 - \gamma)(352 - 320\gamma - 1432\gamma^2 + 860\gamma^3 - 92\gamma^4 - 7\gamma^5 + \gamma^6) \lesseqgtr 0
\end{aligned}$$

The denominator of $\frac{dg_{k_0}}{d\gamma}(N, \gamma)$ being positive, the derivative is of the same sign as its numerator Φ_{k_0}. The numerator is a second degree polynomial in N with $\frac{d^2\Phi_{k_0}}{dN^2} = 2\phi_2 \leq 0$ and so the first derivative of Φ_{k_0} is a monotonically decreasing function of N with $\frac{d\Phi_{k_0}}{dN}(4, \gamma) = -\gamma(480 + 720\gamma + 704\gamma^2 - 2352\gamma^3 + 426\gamma^4 + 47\gamma^5 - 6\gamma^6) \leq 0$. Hence, Φ_{k_0} is a decreasing function of N for $N \geq 4$. For $N = 10$, we have

$$\Phi_{k_0}(10, \gamma) = -(704 + 3808\gamma + 4656\gamma^2 + 18832\gamma^3 - 34164\gamma^4 + 5778\gamma^5 + 639\gamma^6 - 81\gamma^7) < 0$$

So for $N \geq 10$, $\frac{dg_{k_0}}{d\gamma}$ is negative. Let us now calculate the derivative of k_0 with respect to γ

$$\frac{dk_0}{d\gamma} = f'_{k_0} - \frac{g'_{k_0}}{2\sqrt{g_{k_0}}} = \frac{2\sqrt{g_{k_0}}f'_{k_0} - g'_{k_0}}{2\sqrt{g_{k_0}}}$$

When $g'_{k_0} = \frac{dg_{k_0}}{d\gamma}$ is negative, we have $2\sqrt{g_{k_0}}f'_{k_0} + g'_{k_0} < 0$ and so $\frac{dk_0}{d\gamma}$ will be of the opposite sign of $4g_{k_0}f'^2_{k_0} - g'^2_{k_0} = (2\sqrt{g_{k_0}}f'_{k_0} - g'_{k_0})(2\sqrt{g_{k_0}}f'_{k_0} + g'_{k_0})$. To finish the proof that k_0 is a decreasing function of γ we thus need to determine the sign of $4g_{k_0}f'^2_{k_0} - g'^2_{k_0}$.

$$4g_{k_0}f'^2_{k_0} - g'^2_{k_0} = \frac{X_{k_0}}{108\gamma^6(2+\gamma)^4}$$

where

$$\begin{aligned}
X_{k_0} &\equiv \chi_4 N^4 + \chi_3 N^3 + \chi_2 N^2 + \chi_1 N + \chi_0 \\
\chi_4 &\equiv \gamma^6(36 - 4\gamma - \gamma^2)^2 \geq 0 \\
\chi_3 &\equiv 2\gamma^4(36 - 4\gamma - \gamma^2)(352 - 512\gamma + 204\gamma^2 - 36\gamma^3 + 5\gamma^4) \\
\chi_2 &\equiv \gamma^2(34048 + 54272\gamma - 304256\gamma^2 + 282496\gamma^3 - 98880\gamma^4 + 13504\gamma^5 - 264\gamma^6 - 168\gamma^7 + 23\gamma^8) \\
\chi_1 &\equiv 4\gamma(2-\gamma)(13440 - 12992\gamma - 66816\gamma^2 + 78576\gamma^3 - 29008\gamma^4 + 3984\gamma^5 - 78\gamma^6 - 42\gamma^7 + 5\gamma^8) \\
\chi_0 &\equiv 2(2-\gamma)^3(4928 - 11424\gamma - 24192\gamma^2 + 16208\gamma^3 - 2864\gamma^4 + 116\gamma^5 + 22\gamma^6 - 3\gamma^7)
\end{aligned}$$

The denominator being positive, the expression is of the sign as its numerator X_{k_0} which is a fourth degree polynomial in N. To sign this polynomial we differentiate it successively with respect to N. $\frac{d^4X_{k_0}}{dN^4} = 24\chi_4 \geq 0$ and so the third derivative of X_{k_0} is an increasing function of N. $\frac{d^3X_{k_0}}{dN^3}(6,\gamma) = 2\gamma^4(36 - 4\gamma - \gamma^2)(352 - 512\gamma + 636\gamma^2 - 84\gamma^3 - 7\gamma^4) \geq 0$ and so the third derivative is positive for any $N \geq 6$ and the second derivative is increasing with N. $\frac{d^2X_{k_0}}{dN^2}(6,\gamma) = 2\gamma^2(34048 + 54272\gamma + 151936\gamma^2 - 431744\gamma^3 + 506496\gamma^4 - 106304\gamma^5 - 8040\gamma^6 + 2136\gamma^7 + 59\gamma^8) \geq 0$ and so the second derivative is positive for any $N \geq 6$ and the first derivative is increasing with N. $\frac{dX_{k_0}}{dN}(6,\gamma) = 8\gamma(13440 + 31360\gamma + 21088\gamma^2 - 2256\gamma^3 - 180232\gamma^4 + 254216\gamma^5 - 56118\gamma^6 - 3207\gamma^7 + 1070\gamma^8 + 5\gamma^9) \geq 0$ and so the first derivative is positive for $N \geq 6$ and X_{k_0} is an increasing function of N. Finally, $X_{k_0}(6,\gamma) = 2(39424 + 172032\gamma + 112896\gamma^2 - 124288\gamma^3 - 402976\gamma^4 - 682880\gamma^5 + 1421712\gamma^6 - 335560\gamma^7 - 15140\gamma^8 + 6200\gamma^9 - 75\gamma^10) > 0$ and so X_{k_0} is positive for any $N \geq 6$ and $\gamma > 0$ and so $4g_{k_0}f'^2_{k_0} - g'^2_{k_0} > 0$ and hence k_0 is a decreasing function of γ for $N \geq 10$.

For $N < 10$, $\frac{dg_{k_0}}{d\gamma}$ changes sign with γ. When $\frac{dg_{k_0}}{d\gamma} \leq 0$, the above proof applies (for $N \geq 6$). When $\frac{dg_{k_0}}{d\gamma} > 0$, g_{k_0} is an increasing function of γ and $-\sqrt{g_{k_0}}$ is a decreasing function of γ. Hence $k_0 = f_{k_0}(N,\gamma) - \sqrt{g_{k_0}(N,\gamma)}$ is a decreasing function of γ. $\square$

We have therefore shown that, for any $N \geq 6$ and $\gamma > 0$, k_0 is a decreasing function of γ, and so evaluating k_0 at $\gamma = 1$ gives a lower bound for k_0:

$$k_0(\gamma = 1) = \frac{1}{18}\left[10N + 28 - \sqrt{2(14N^2 - 62N - 49)}\right] \geq \frac{5 - \sqrt{7}}{9}N$$

To finish the proof for $N < 6$, we check that $k_0(N,\gamma) > N$ for $N = 1, 2 \ldots 5$. Hence we have shown that, for any N and any γ, $k_0 \geq \frac{5-\sqrt{7}}{9}N$ and so there will be at most four CUs in equilibrium.

Furthermore, we can note that $k_0(\gamma = 1) > \frac{N}{3}$ for $N \leq 31$ and so we know that for $N \leq 31$ there will be at most three CUs in equilibrium.

As k_0 is a decreasing function of γ and $k_0(\gamma = 1) < \frac{N}{3}$ for $N > 31$, there exists a unique $\gamma_3 \in (0, 1)$ for which $k_0 = \frac{N}{3}$. This γ_3 is a complicated function of N, but $\underline{\gamma_3} = \frac{1}{2}(13 - \sqrt{129}) \approx 0.821092$ is a lower bound for γ_3, because $k_0(\underline{\gamma_3}) \geq \frac{N}{3}$ (with equality for $N = +\infty$). Hence we also know that for $N > 31$ and $\gamma \in [0, \frac{1}{2}(13 - \sqrt{129})]$ there will be at most three CUs in equilibrium.

b) $\boldsymbol{k_0}$ is such that a CU of size $\boldsymbol{k_0}$ is not constrained by Article XXIV, but a CU of size $\boldsymbol{k_0 - 1}$ is constrained by Article XXIV

If this case occurs, we have $W_u(k, C) - W_c(k-1, C') \geq W_c(k, C) - W_c(k-1, C')$ and so the derivations from case a) above provide also a lower bound for k_0 is this case.

c) $\boldsymbol{k_0}$ is such that neither a CU of size $\boldsymbol{k_0}$ nor of size $\boldsymbol{k_0 - 1}$ are constrained by Article XXIV

This case corresponds to the case without the Article XXIV constraint analyzed by Yi (1996) who shows that there will be at most three CUs in equilibrium. □

Proof of Proposition 4. We assume that the equilibrium CU structure consists of at most two blocs: a bloc of size k which forms first and a bloc of size $N - k$. From Lemma 7 we know that the two blocs will necessarily be asymmetric with the larger bloc forming first and so we have $k > N - k$. (Note that we allow the small bloc to be empty ($k = N$) in which case there is only one bloc in equilibrium.)

The aim of this proof is to determine which bloc will be bound by Article XXIV and when. We already know that, for $\gamma \leq \gamma_{N\infty}$, CUs of any size are bound by Article XXIV so both the small and the large CUs will be bound. If $k < N$, then the range of γ for which both equilibrium CUs will be constrained will be larger. We also know that, as $k > \frac{N}{2}$, the large CU will not be constrained for $\gamma > \gamma_{\frac{N}{2}}$. Hence both blocs will be constrained on a subrange of $(0, \gamma_{\frac{N}{2}})$.[2]

To determine exactly which bloc will be constrained and when, we need to determine the equilibrium size of the two blocs. The large CU (the first bloc to form) is choosing its size k to maximize its welfare knowing that the second bloc will be formed by the remaining countries. The optimization problem is thus

$$\underset{k}{\operatorname{argmax}}\, W(k, \{k, N-k\})$$

To solve this optimization problem, we need to calculate the first derivative of the welfare function of the large CU $W(k, \{k, N-k\})$ with respect to its size k. But the welfare function

[2]From Lemma 7 we know that for $\gamma = 0$ there will be only one CU in equilibrium and so Article XXIV will have no hold.

$W(k, \{k, N-k\})$ changes depending on which of the blocs is constrained. So to determine the equilibrium size of the two blocs, we need to know whether they are constrained or not. To solve this problem, we therefore have to determine the equilibrium size of the blocs making assumptions on whether they are constrained or not, and then we have to check that the obtained equilibrium sizes do not contradict our assumptions.

Hence the proof proceeds in three steps. First, we solve for the equilibrium size of the two blocs assuming that they are both constrained and we determine the range of parameters for which the obtained equilibrium sizes are indeed such that the two blocs are constrained. Second, we solve for the equilibrium size of the two blocs assuming that only the small bloc is constrained and we show that, on the remainder of the parameter range, the equilibrium sizes are such that only the small bloc is constrained. Finally, to terminate the proof, we solve for the equilibrium size of the two blocs assuming that neither of them is constrained and we show that the equilibrium sizes are such that the small bloc would necessarily be constrained. Hence we can conclude that the case in which neither bloc is constrained never arises in equilibrium with Article XXIV.

1) Assume that both blocs are constrained by Article XXIV (CC):
Making use of (6), (7) and (8) with $\tau_c(k) = \tau_c(N-k) = \tau(1)$, we can calculate the first derivative of the welfare function of the large bloc with respect to its size

$$\frac{dW_{cc}(k, \{k, N-k\})}{dk} = \frac{2+\gamma}{2\Gamma(N)^2 D(1)^2} \left[\lambda_0(N,\gamma) - 2\gamma\lambda_1(N,\gamma)k\right] \tag{D.4}$$

with

$$\lambda_0(N,\gamma) \equiv \Gamma(0)(28 - 20\gamma + 5\gamma^2 + \gamma^3) + 4\gamma(30 - 29\gamma + 13\gamma^2)N + \gamma^2(34 - 41\gamma + \gamma^2)N^2$$
$$\lambda_1(N,\gamma) \equiv 64 - 60\gamma + 32\gamma^2 - \gamma^3 + \gamma(22 - 27\gamma + \gamma^2)N$$

When $\gamma = 0$, the derivative is strictly positive and independent of k: $\frac{dW(k,\{k,N-k\})}{dk} = \frac{7}{72} > 0$. The optimal size of the large bloc is thus $k_{cc}^{opt} = N$.
When $0 < \gamma \leq \gamma_{\frac{N}{2}}$, $\lambda_0(N,\gamma) > 0$ and $\lambda_1(N,\gamma) > 0$ and from setting (D.4) equal to zero we have

$$k_{cc}^{opt}(N,\gamma) = \frac{\lambda_0(N,\gamma)}{2\gamma\lambda_1(N,\gamma)} \tag{D.5}$$

We can note that $\frac{k_{cc}^{opt}}{N}$ is a monotonically decreasing function of N.

$$\frac{d}{dN}\left[\frac{k_{cc}^{opt}}{N}(N,\gamma)\right] = -\frac{\lambda_2(N,\gamma)}{2\gamma N^2 \lambda_1(N,\gamma)^2} < 0$$

with $\lambda_2(N,\gamma) \equiv \xi_2 N^2 + \xi_1 N + \xi_0$ where

$$\xi_2 \equiv \gamma^2(464 - 1128\gamma + 784\gamma^2 - 114\gamma^3 - 21\gamma^4 + \gamma^5) \geq 0 \text{ for } \gamma \in [0, \gamma_{\frac{N}{2}}]$$
$$\xi_1 \equiv 2\gamma\Gamma(0)(22 - 27\gamma + \gamma^2)(28 - 20\gamma + 5\gamma^2 + \gamma^3) \geq 0 \text{ for } \gamma \in [0, \gamma_{\frac{N}{2}}]$$
$$\xi_0 \equiv \Gamma(0)(64 - 60\gamma + 32\gamma^2 - \gamma^3)(28 - 20\gamma + 5\gamma^2 + \gamma^3) > 0 \text{ for } \gamma \in [0, \gamma_{\frac{N}{2}}]$$

And so for any N, $\frac{k_{cc}^{opt}}{N} \geq \lim_{N\to\infty} \frac{k_{cc}^{opt}}{N} = \frac{34 - 41\gamma + \gamma^2}{44 - 54\gamma + 2\gamma^2} \geq \frac{17}{22}$. Hence, if the size of the large bloc is k_{cc}^{opt}, we know that the size of the small bloc is smaller than $\frac{5}{22}N < k^{**}(N,1)$ and so the small bloc is necessarily constrained by Article XXIV.

The question now is whether the size of the large bloc k_{cc}^{opt} (which we obtained assuming that both CUs are bound by Article XXIV) is such that the large bloc is also constrained. In other words, is k_{cc}^{opt} the relevant solution for the equilibrium size of the large CU? We show that this is the case for a subrange of γ. To show this we determine when $k_{cc}^{opt} \leq k^{**}$. Define $\Delta_k \equiv k^{**} - k_{cc}^{opt}$. After simplifications we obtain

$$\Delta_k(N,\gamma) = \frac{\delta_k(N,\gamma)}{2\gamma(2+\gamma)\lambda_1(N,\gamma)} \tag{D.6}$$

with

$$\begin{aligned}\delta_k(N,\gamma) \equiv\ & \Gamma(0)(200 - 612\gamma + 562\gamma^2 - 263\gamma^3 + 37\gamma^4 - \gamma^5) \\ & + 2\gamma(160 - 568\gamma + 592\gamma^2 - 266\gamma^3 + 36\gamma^4 - \gamma^5)N \\ & \gamma^2(64 - 224\gamma + 202\gamma^2 - 33\gamma^3 + \gamma^4)N^2\end{aligned}$$

$\Delta_k(N,\gamma)$ is a continuous function of γ for $\gamma \in (0, \gamma_{\frac{N}{2}}]$. It is easy to show that it is a monotonically decreasing function of γ on this range:

$$\frac{d\Delta_k(N,\gamma)}{d\gamma} = -\frac{\mu_k(N,\gamma)}{2\gamma^2(2+\gamma)^2\lambda_1(N,\gamma)^2} < 0$$

where $\mu_k(N,\gamma) \equiv \mu_0 + \mu_1 N + \mu_2 N^2 + \mu_3 N^3$ with

$$\begin{aligned}\mu_0 \equiv\ & 51200 - 44800\gamma - 137664\gamma^2 + 315136\gamma^3 - 313552\gamma^4 + 167568\gamma^5 - 46388\gamma^6 + 2712\gamma^7 \\ & + 821\gamma^8 - 60\gamma^9 + \gamma^{10} > 0 \text{ for } \gamma \in (0, \gamma_{\frac{N}{2}}] \\ \mu_1 \equiv\ & \gamma(35200 + 26432\gamma - 249408\gamma^2 + 395968\gamma^3 - 288040\gamma^4 + 102772\gamma^5 - 8240\gamma^6 - 2146\gamma^7 \\ & + 170\gamma^8 - 3\gamma^9) > 0 \text{ for } \gamma \in (0, \gamma_{\frac{N}{2}}] \\ \mu_2 \equiv\ & \gamma^2(5888 + 36864\gamma - 129600\gamma^2 + 148256\gamma^3 - 73440\gamma^4 + 8264\gamma^5 + 1852\gamma^6 - 160\gamma^7 \\ & + 3\gamma^8) > 0 \text{ for } \gamma \in (0, \gamma_{\frac{N}{2}}] \\ \mu_3 \equiv\ & \gamma^4(7808 - 20976\gamma + 16612\gamma^2 - 2736\gamma^3 - 527\gamma^4 + 50\gamma^5 - \gamma^6) > 0 \text{ for } \gamma \in (0, \gamma_{\frac{N}{2}}]\end{aligned}$$

Furthermore, we have

$$\Delta_k(N, \frac{466}{1000}) = \frac{642115906555069N^2 - 20005524915931138N + 286625284171126069}{287289000(561247837N + 5360974663)}$$
$$> 0 \text{ for } N \geq 1$$

$$\Delta_k(N, \frac{476}{1000}) = -\frac{20100163461469N^2 + 720384736178562N - 3547369240077531}{36830500(69723409N + 665352841)}$$
$$< 0 \text{ for } N \geq 5$$

So by the intermediate value theorem, we know that there exists a unique $\hat{\gamma}(N) \in \left(\frac{466}{1000}, \frac{476}{1000}\right)$ such that $\Delta_k[N, \hat{\gamma}(N)] = 0$. Hence for $\gamma \leq \hat{\gamma}(N)$ we have $k_{cc}^{opt} \leq k^{**}$ and for $\gamma > \hat{\gamma}(N)$ we have $k_{cc}^{opt} > k^{**}$. Therefore for $\gamma \leq \hat{\gamma}(N)$ both CUs are well constrained by Article XXIV and k_{cc}^{opt} is the relevant solution for the equilibrium size of the large CU on this interval.[3]

2) Assume that only the small bloc is constrained by Article XXIV (UC):
The solution k_{cc}^{opt} obtained above assuming that both CUs are constrained by Article XXIV is such that for $\gamma > \hat{\gamma}(N)$ the CU of size k_{cc}^{opt} would not be constrained by Article XXIV. Therefore, for $\gamma > \hat{\gamma}(N)$, k_{cc}^{opt} cannot be the relevant solution for the size of the large CU. We now assume that only the small CU is constrained by Article XXIV and solve for the equilibrium size of the large union k_{uc}^{opt}. In the remainder of the proof we are assuming $\gamma > \hat{\gamma}(N)$. Given that $\hat{\gamma}(N) > 0.466$, we prove all the results for $\gamma > 0.466$ which proves them for $\gamma > \hat{\gamma}(N)$.

Again making use of (6), (7) and (8), but now with the large bloc imposing $\tau_c(k) = \tau(k) \leq \tau(1)$ and the small bloc imposing $\tau_c(N-k) = \tau(1)$, we can calculate the first derivative of the welfare function of the large bloc with respect to its size

$$\frac{dW_{uc}(k, \{k, N-k\})}{dk} = \frac{H_{uc}(k, N, \gamma)}{2\Gamma(N)^2 D(1)^2 D(k)^2} \tag{D.7}$$

with

$$H_{uc}(k, N, \gamma) \equiv \eta_6(N,\gamma)k^6 + \eta_5(N,\gamma)k^5 + \eta_4(N,\gamma)k^4 + \eta_3(N,\gamma)k^3 + \eta_2(N,\gamma)k^2 + \eta_1(N,\gamma)k + \eta_0$$

and

$$\eta_6(N,\gamma) \equiv -24\gamma^6(2+\gamma)^2$$

[3]For $N \geq 9$, we can even refine the range of $\hat{\gamma}(N)$ because we have

$$\Delta_k(N, \frac{470}{1000}) = -\frac{7855687771N^2 + 1946145300258N - 16803834658029}{2321800(4479523N + 42764977)} < 0 \text{ for } N \geq 9$$

and so we know that $\hat{\gamma}(N) \in [\frac{466}{1000}, \frac{470}{1000}]$.

$$
\begin{aligned}
\eta_5(N,\gamma) \equiv\; & -8\gamma^5(2+\gamma)\left[\gamma(10-21\gamma-3\gamma^2)N+80-44\gamma+4\gamma^2+3\gamma^3\right]\\
\eta_4(N,\gamma) \equiv\; & 2\gamma^4\left[-\gamma^2(28-188\gamma+143\gamma^2+78\gamma^3+3\gamma^4)N^2\right.\\
& +2\gamma(-448+1032\gamma-228\gamma^2-134\gamma^3+53\gamma^4+3\gamma^5)N\\
& \left.-3648+3648\gamma-912\gamma^2-416\gamma^3+260\gamma^4-28\gamma^5-3\gamma^6\right]\\
\eta_3(N,\gamma) \equiv\; & 4\gamma^3\left[\gamma^3(1-\gamma)(4+28\gamma-55\gamma^2-9\gamma^3)N^3\right.\\
& -\gamma^2(144-992\gamma+1272\gamma^2-420\gamma^3-15\gamma^4+23\gamma^5)N^2\\
& +\gamma(-2128+6496\gamma-5736\gamma^2+1904\gamma^3-5\gamma^4-130\gamma^5+19\gamma^6)N\\
& \left.+(2-\gamma)(-2880+4048\gamma-2320\gamma^2+408\gamma^3+132\gamma^4-59\gamma^5+5\gamma^6)\right]\\
\eta_2(N,\gamma) \equiv\; & 2\gamma^2\left[\gamma^4(1-\gamma)(52-88\gamma+23\gamma^2+15\gamma^3)N^4\right.\\
& +4\gamma^3(136-254\gamma+31\gamma^2+113\gamma^3-46\gamma^4+4\gamma^5)N^3\\
& +\gamma^2(752+2624\gamma-7900\gamma^2+6612\gamma^3-2166\gamma^4+156\gamma^5+35\gamma^6)N^2\\
& 2\gamma(2-\gamma)(-1872+6696\gamma-7444\gamma^2+3742\gamma^3-736\gamma^4-53\gamma^5+29\gamma^6)N\\
& \left.2(2-\gamma)^2(-2432+4176\gamma-3256\gamma^2+1184\gamma^3-112\gamma^4-49\gamma^5+11\gamma^6)\right]\\
\eta_1(N,\gamma) \equiv\; & 4\gamma(2-\gamma)^2\left[\gamma^4(52-88\gamma+23\gamma^2+15\gamma^3)N^4\right.\\
& +\gamma^3(508-788\gamma+244\gamma^2+84\gamma^3-43\gamma^4)N^3\\
& +8\gamma^2(3-\gamma)(2+\gamma)(33-49\gamma+26\gamma^2-5\gamma^3)N^2\\
& +\gamma(1264-720\gamma-1008\gamma^2+1452\gamma^3-731\gamma^4+158\gamma^5-11\gamma^6)N\\
& \left.-(2-\gamma)(3-\gamma)(160-304\gamma+288\gamma^2-120\gamma^3+22\gamma^4+\gamma^5)\right]\\
\eta_0(N,\gamma) \equiv\; & (2-\gamma)^2\left[\gamma^4(272-536\gamma+215\gamma^2+53\gamma^3-30\gamma^4)N^4\right.\\
& +4\gamma^3(648-1286\gamma+747\gamma^2-8\gamma^3-115\gamma^4+26\gamma^5)N^3\\
& +2\gamma^2(4328-8824\gamma+6382\gamma^2-1076\gamma^3-901\gamma^4+471\gamma^5-66\gamma^6)N^2\\
& +4\gamma(2-\gamma)(2+\gamma)(3-\gamma)(236-428\gamma+335\gamma^2-125\gamma^3+18\gamma^4)N\\
& \left.+(2-\gamma)^2(3-\gamma)(336-352\gamma+40\gamma^2+128\gamma^3-83\gamma^4+14\gamma^5)\right]
\end{aligned}
$$

The optimal size of the large CU k_{uc}^{opt} is given by setting (D.7) equal to zero. It is hard to find a closed-form solution of this polynomial equation of degree 6 in k, we can however provide a lower bound for k_{uc}^{opt} by studying further the derivative of the welfare function (D.7).

The small CU being constrained by Article XXIV, we are interested in the variations of the derivative (D.7) for $N-k \leq k^{**} \Leftrightarrow k \geq N-k^{**}$. We show that the derivative (D.7) is strictly positive for $k \in [N-k^{**}, \frac{78}{100}N]$ and so we will have shown that $k_{uc}^{opt} \geq \frac{78}{100}N$.

The denominator of the derivative (D.7) is strictly positive and so the derivative has the same sign as its numerator $H_{uc}(k,N,\gamma)$.

Lemma. *$H_{uc}(k, N, \gamma)$ changes sign only once in the interval $[N - k^{**}, N]$. It is initially an increasing function of k and then it becomes a decreasing function with $H_{uc}(N - k^{**}, N, \gamma) > 0$ and $H_{uc}(N, N, \gamma) < 0$.*

Proof. $H_{uc}(k, N, \gamma)$ is a sixth degree polynomial in k. To determine the variations and the sign of H_{uc} we differentiate successively with respect to k. The sixth derivative of H_{uc} with respect to k is

$$\frac{\partial^6 H_{uc}}{\partial k^6}(k, N, \gamma) = -17280\gamma^6(2+\gamma)^2 < 0 \text{ for } \gamma \geq 0.466$$

And so the fifth derivative is a decreasing function of k. Evaluating the fifth derivative at the lower bound of the interval of interest $k \in [N - k^{**}, N]$ gives

$$\frac{\partial^5 H_{uc}}{\partial k^5}(N - k^{**}, N, \gamma) = 1920\gamma^5(2+\gamma)\Big[\gamma \underbrace{(4 - 21\gamma + 6\gamma^2)}_{\leq 0 \text{ for } \gamma \geq 0.466} N - 2 \underbrace{(2 + 25\gamma - 17\gamma^2 + 3\gamma^3)}_{>0 \text{ for } \gamma \in [0,1]}\Big]$$

So $\frac{\partial^5 H_{uc}}{\partial k^5}(N - k^{**}, N, \gamma) < 0$ for $\gamma \geq 0.466$ and so the fifth derivative is negative for any $k \in [N - k^{**}, N]$ and so the fourth derivative is decreasing on this range. Evaluating the fourth derivative at the lower bound yields

$$\begin{aligned}\frac{\partial^4 H_{uc}}{\partial k^4}(N - k^{**}, N, \gamma) = & -96\gamma^4 \Bigg[\gamma^2 \underbrace{(4 - 164\gamma + 509\gamma^2 - 276\gamma^3 + 39\gamma^4)}_{>0 \text{ for } \gamma \geq 0.466} N^2 \\ & + 2\gamma \underbrace{(-16 - 56\gamma + 1204\gamma^2 - 1108\gamma^3 + 361\gamma^4 - 39\gamma^5)}_{>0 \text{ for } \gamma \geq 0.466} N \\ & \underbrace{64 + 576\gamma + 2216\gamma^2 - 3552\gamma^3 + 1890\gamma^4 - 446\gamma^5 + 39\gamma^6}_{>0 \text{ for } \gamma \geq 0.466}\Bigg] < 0\end{aligned}$$

The fourth derivative is decreasing with k and negative at the lower bound of the interval considered and so it is negative on the entire interval $k \in [N - k^{**}, N]$ and so the third derivative is decreasing on this range. Evaluating the third derivative at lower bound $N - k^{**}$ is inconclusive as it can be either positive or negative depending on the parameters. However, evaluating the third derivative at the upper bound the upper bound of the interval ($k = N$) yields

$$\frac{\partial^3 H_{uc}}{\partial k^3}(N,N,\gamma) = 24\gamma^3 \Bigg[\gamma^3 \underbrace{(-932+560\gamma+51\gamma^2-50\gamma^3+3\gamma^4)}_{<0 \text{ for } \gamma\geq 0.466} N^3$$
$$\underbrace{-\gamma^2(5136-5280\gamma+1464\gamma^2+316\gamma^3-167\gamma^4+11\gamma^5)}_{<0 \text{ for } \gamma\geq 0.466} N^2$$
$$+\underbrace{\gamma(-9424+13792\gamma-7560\gamma^2+1072\gamma^3+515\gamma^4-186\gamma^5+13\gamma^6)}_{<0 \text{ for } \gamma\geq 0.466} N$$
$$\underbrace{-(2-\gamma)(2880-4048\gamma+2320\gamma^2-408\gamma^3-132\gamma^4+59\gamma^5-5\gamma^6)}_{<0 \text{ for } \gamma\geq 0.466}\Bigg] < 0$$

Hence we know that the second derivative is either monotonically decreasing or initially increasing and then decreasing function for $k \in [N-k^{**}, N]$. Evaluating the second derivative at the upper bound $k = N$ yields

$$\frac{\partial^2 H_{uc}}{\partial k^2}(N,N,\gamma) = 4\gamma^2 \Bigg[\gamma^4 \underbrace{(-1612+1692\gamma-345\gamma^2-80\gamma^3+21\gamma^4)}_{<0 \text{ for } \gamma\geq 0.466} N^4$$
$$\underbrace{-2\gamma^3(6048-8820\gamma+4402\gamma^2-482\gamma^3-211\gamma^4+43\gamma^5)}_{<0 \text{ for } \gamma\geq 0.466} N^3$$
$$+\gamma^2 \underbrace{(-33904+63488\gamma-47788\gamma^2+15540\gamma^3-636\gamma^4-792\gamma^5+131\gamma^6)}_{<0 \text{ for } \gamma\geq 0.466} N^2$$
$$+4\gamma(2-\gamma)\underbrace{(-5256+9420\gamma-7202\gamma^2+2483\gamma^3-170\gamma^4-115\gamma^5+22\gamma^6)}_{<0 \text{ for } \gamma\geq 0.466} N$$
$$2(2-\gamma)^2 \underbrace{(-2432+4176\gamma-3256\gamma^2+1184\gamma^3-112\gamma^4-49\gamma^5+11\gamma^6)}_{<0 \text{ for } \gamma\geq 0.466}\Bigg] < 0$$

So the second derivative is negative at $k = N$. At $k = N - k^{**}$, the second derivative can be either positive or negative depending on the parameters. Furthermore, we show that at $k = N - k^{**}$, the second derivative is greater than the third derivative:

$$\frac{\partial^2 H_{uc}}{\partial k^2}(N-k^{**},N,\gamma) - \frac{\partial^3 H_{uc}}{\partial k^3}(N-k^{**},N,\gamma) = \frac{\gamma^2}{(2+\gamma)^2}\left[\nu_4 N^4 + \nu_3 N^3 + \nu_2 N^2 + \nu_1 N + \nu_0\right]$$

with

$$\nu_4 \equiv \gamma^4(848-2848\gamma+5144\gamma^2+2328\gamma^3-17867\gamma^4+20080\gamma^5-9082\gamma^6+1776\gamma^7-123\gamma^8)$$

$$\nu_3 \equiv 4\gamma^3(2112 - 5856\gamma + 14944\gamma^2 - 16968\gamma^3 - 18000\gamma^4 + 53844\gamma^5 - 42587\gamma^6 + 14649\gamma^7 - 2245\gamma^8 + 123\gamma^9)$$

$$\nu_2 \equiv 2\gamma^2(16384 - 40192\gamma + 82656\gamma^2 - 202912\gamma^3 + 139360\gamma^4 + 276576\gamma^5 - 432602\gamma^6 + 242446\gamma^7 - 65015\gamma^8 + 8142\gamma^9 - 369\gamma^{10})$$

$$\nu_1 \equiv 4\gamma(13312 - 37248\gamma + 46208\gamma^2 - 131040\gamma^3 + 312128\gamma^4 - 17544\gamma^5 - 354296\gamma^6 + 330878\gamma^7 - 138832\gamma^8 + 30150\gamma^9 - 3183\gamma^{10} + 123\gamma^{11})$$

$$\nu_0 \equiv (2-\gamma)(14336 - 46080\gamma + 43520\gamma^2 - 13568\gamma^3 + 467328\gamma^4 - 200256\gamma^5 - 325568\gamma^6 + 358112\gamma^7 - 154160\gamma^8 + 33272\gamma^9 - 3406\gamma^{10} + 123\gamma^{11})$$

All the polynomial functions ν_i are positive for $\gamma \geq 0.466$ and so we have $\frac{\partial^2 H_{uc}}{\partial k^2}(N - k^{**}, N, \gamma) \geq \frac{\partial^3 H_{uc}}{\partial k^3}(N - k^{**}, N, \gamma)$. Thus when the third derivative is positive at $N - k^{**}$, the second derivative is also necessarily positive at this point. Hence we know that the second derivative is either monotonically decreasing and negative on the whole interval $[N - k^{**}, N]$, or it is monotonically decreasing taking initially positive values and then negative values, or it is initially increasing and positive and then decreasing and negative for $k = N$. Consequently, the first derivative is either monotonically decreasing or initially increasing and then decreasing. We now sign the first derivative at the bounds of the interval $[N - k^{**}, N]$. The first derivative is negative at the upper bound $k = N$

$$\begin{aligned}\frac{\partial H_{uc}}{\partial k}(N, N, \gamma) = & -4\gamma(3-\gamma)\Gamma(N)^2\Big[2\gamma^3(2-\gamma)(7+\gamma)(4-3\gamma)N^3 \\ & + 2\gamma^2\underbrace{\left(288 - 404\gamma + 202\gamma^2 - 26\gamma^3 - 7\gamma^4\right)}_{>0 \text{ for } \gamma\geq 0.466}N^2 \\ & + \gamma\underbrace{\left(880 - 1536\gamma + 1216\gamma^2 - 436\gamma^3 + 49\gamma^4 + 9\gamma^5\right)}_{>0 \text{ for } \gamma\geq 0.466}N \\ & (2-\gamma)\underbrace{\left(160 - 304\gamma + 288\gamma^2 - 120\gamma^3 + 22\gamma^4 + \gamma^5\right)}_{>0 \text{ for } \gamma\geq 0.466}\Big] < 0\end{aligned}$$

The first derivative is positive at the lower bound $k = N - k^{**}$ for $N \geq 5$

$$\frac{\partial H_{uc}}{\partial k}(N - k^{**}, N, \gamma) = \frac{\gamma\Gamma(N)}{(2+\gamma)^3}\left[\iota_4 N^4 + \iota_3 N^3 + \iota_2 N^2 + \iota_1 N + \iota_0\right]$$

with

$$\iota_4 \equiv \gamma^4(2-\gamma)(3-\gamma)\underbrace{(2-7\gamma+\gamma^2)}_{<0 \text{ for } \gamma\geq 0.466}\underbrace{(-64+84\gamma-92\gamma^2+51\gamma^3+67\gamma^4-93\gamma^5+15\gamma^6)}_{<0 \text{ for } \gamma\geq 0.466} > 0$$

$$\iota_3 \equiv \gamma^3(-3392+39872\gamma-84848\gamma^2+110032\gamma^3-104396\gamma^4+29660\gamma^5+60651\gamma^6-75039\gamma^7 + 37821\gamma^8-9713\gamma^9+1228\gamma^{10}-60\gamma^{11}) > 0 \text{ for } \gamma \geq 0.466$$

$$\iota_2 \equiv \gamma^2(7168+133248\gamma-325312\gamma^2+427296\gamma^3-510288\gamma^4+307896\gamma^5+113772\gamma^6-320946\gamma^7 + 232453\gamma^8-88084\gamma^9+18621\gamma^{10}-2046\gamma^{11}+90\gamma^{12})$$

$$\iota_1 \equiv \gamma(2-\gamma)(24576+99328\gamma-237824\gamma^2+233088\gamma^3-364256\gamma^4+256688\gamma^5+60064\gamma^6 - 218392\gamma^7+159374\gamma^8-60163\gamma^9+12655\gamma^{10}-1380\gamma^{11}+60\gamma^{12})$$

$$\iota_0 \equiv (2-\gamma)^2(13312+27136\gamma-73472\gamma^2+32896\gamma^3-98240\gamma^4+80192\gamma^5+11200\gamma^6-56056\gamma^7 + 40968\gamma^8-15390\gamma^9+3223\gamma^{10}-349\gamma^{11}+15\gamma^{12})$$

The polynomial functions ι_4 and ι_3 are positive for $\gamma \geq 0.466$ while the polynomial functions ι_2, ι_1 and ι_0 change sign on $\gamma \in [0.466, 1]$ (they become negative for γ close to 1). Easy successive differentiation with respect to N of $\iota_4 N^4 + \iota_3 N^3 + \iota_2 N^2 + \iota_1 N + \iota_0$ shows however that $\frac{\partial H_{uc}}{\partial k}(N-k^{**}, N, \gamma) > 06$ for $N \geq 5$.

Hence we know that the numerator the of derivative of welfare $H_{uc}(k, N, \gamma)$ is an initially increasing and then decreasing function of k in the interval $[N-k^{**}, N]$. Let us now sign the numerator at the bounds of this interval. At the lower bound $k = N - k^{**}$ we have

$$H_{uc}(N-k^{**}, N, \gamma) = \frac{\Gamma(N)^2}{2(2+\gamma)^3}\Big[\upsilon_4 N^4 + \upsilon_3 N^3 + \upsilon_2 N^2 + \upsilon_1 N + \upsilon_0\Big]$$

with

$$\upsilon_4 \equiv \gamma^4(1-\gamma)(3-\gamma)(2-7\gamma+\gamma^2)^2(32-32\gamma+24\gamma^2-28\gamma^3+18\gamma^4-3\gamma^5)$$

$$\upsilon_3 \equiv 2\gamma^3(3-\gamma)(2-7\gamma+\gamma^2)(-64-1376\gamma+3280\gamma^2-3344\gamma^3+2940\gamma^4-2674\gamma^5+1661\gamma^6 - 581\gamma^7+98\gamma^8-6\gamma^9)$$

$$\upsilon_2 \equiv 2\gamma^2(-8704+21248\gamma+128928\gamma^2-408048\gamma^3+547696\gamma^4-533016\gamma^5+478074\gamma^6 - 368855\gamma^7+209438\gamma^8-80732\gamma^9+20168\gamma^{10}-3080\gamma^{11}+258\gamma^{12}-9\gamma^{13})$$

$$\upsilon_1 \equiv 2\gamma(2-\gamma)(-10752+54144\gamma+63616\gamma^2-283296\gamma^3+359360\gamma^4-338872\gamma^5+309960\gamma^6 - 245574\gamma^7+141350\gamma^8-54800\gamma^9+13717\gamma^{10}-2091\gamma^{11}+174\gamma^{12}-6\gamma^{13})$$

$$\upsilon_0 \equiv (2-\gamma)^2(-6400+55168\gamma+18496\gamma^2-157088\gamma^3+172528\gamma^4-158968\gamma^5+151676\gamma^6 - 123126\gamma^7+71490\gamma^8-27836\gamma^9+6986\gamma^{10}-1064\gamma^{11}+88\gamma^{12}-3\gamma^{13})$$

All the coefficients υ_i are positive for $\gamma \geq 0.466$ and so the numerator $H_{uc}(N-k^{**}, N, \gamma) > 0$.

At the upper bound $k = N$ we have

$$H_{uc}(N, N, \gamma) = -(3-\gamma)\Gamma(N)^2\Big[\bar{\upsilon}_4 N^4 + \bar{\upsilon}_3 N^3 + \bar{\upsilon}_2 N^2 + \bar{\upsilon}_1 N + \bar{\upsilon}_0\Big]$$

with

$$\begin{aligned}
\bar{\upsilon}_4 &\equiv 4\gamma^4(4-3\gamma)^2 > 0 \text{ for } \gamma \geq 0.466 \\
\bar{\upsilon}_3 &\equiv 4\gamma^3(10-\gamma)(4-3\gamma)(2-2\gamma+\gamma^2) > 0 \text{ for } \gamma \geq 0.466 \\
\bar{\upsilon}_2 &\equiv \gamma^2(160-640\gamma+1220\gamma^2-972\gamma^3+343\gamma^4-38\gamma^5) > 0 \text{ for } \gamma \geq 0.466 \\
\bar{\upsilon}_1 &\equiv 2\gamma(2-\gamma)(4-\gamma)(-72+52\gamma+46\gamma^2-59\gamma^3+20\gamma^4) < 0 \text{ for } \gamma \geq 0.466 \\
\bar{\upsilon}_0 &\equiv -(2-\gamma)^2(336-352\gamma+40\gamma^2+128\gamma^3-83\gamma^4+14\gamma^5) < 0 \text{ for } \gamma \geq 0.466
\end{aligned}$$

The coefficients $\bar{\upsilon}_4$, $\bar{\upsilon}_3$ and $\bar{\upsilon}_2$ are strictly positive for $\gamma \geq 0.466$ while $\bar{\upsilon}_1$ and $\bar{\upsilon}_0$ are negative. Easy successive differentiation with respect to N of $\bar{\upsilon}_4 N^4 + \bar{\upsilon}_3 N^3 + \bar{\upsilon}_2 N^2 + \bar{\upsilon}_1 N + \bar{\upsilon}_0$ shows that $H_{uc}(N, N, \gamma) < 0$ for $N \geq 5$ and $\gamma \geq 0.466$.

Hence we know that $H_{uc}(k, N, \gamma)$ changes sign only once in the interval $[N - k^{**}, N]$. It is initially positive and increasing function and then it becomes a decreasing function and negative. $\square$

The Lemma above helps up to determine a lower bound k_{uc}^{opt} for which $H_{uc}(k_{uc}^{opt}, N, \gamma) = 0$. By showing that $H_{uc}(k, N, \gamma)$ is positive at $k = \frac{78}{100}N$ we will have shown that $k_{uc}^{opt} \geq \frac{78}{100}N$.

$$H_{uc}(\frac{78}{100}N, N, \gamma) = \frac{\tilde{\Xi}(N, \gamma)}{1953125000} > 0 \text{ for } \gamma > 0.466$$

where $\tilde{\Xi}(N, \gamma) \equiv \tilde{\xi}_6 N^6 + \tilde{\xi}_5 N^5 + \tilde{\xi}_4 N^4 + \tilde{\xi}_3 N^3 + \tilde{\xi}_2 N^2 + \tilde{\xi}_1 N + \tilde{\xi}_0$ with

$$\begin{aligned}
\tilde{\xi}_6 \equiv\ & 1521\gamma^6(-22697792 + 85616608\gamma - 91674048\gamma^2 + 34374100\gamma^3 - 4351875\gamma^4) \\
& \geq 0 \text{ for } \gamma \in [0.466, 1] \\
\tilde{\xi}_5 \equiv\ & 1950\gamma^5(4696640 + 446727728\gamma - 972099424\gamma^2 + 752546940\gamma^3 - 239698898\gamma^4 \\
& + 27095175\gamma^5) > 0 \text{ for } \gamma \in [0.466, 1] \\
\tilde{\xi}_4 \equiv\ & 625\gamma^4(5008964032 - 3693184832\gamma - 10736486592\gamma^2 + 17851076144\gamma^3 - 10604004840\gamma^4 \\
& + 2793891652\gamma^5 - 274146723\gamma^6) > 0 \text{ for } \gamma \in [0.466, 1] \\
\tilde{\xi}_3 \equiv\ & 62500\gamma^3\Gamma(0)(157675680 - 214745888\gamma - 37216280\gamma^2 + 225886252\gamma^3 - 159352692\gamma^4 \\
& + 45324529\gamma^5 - 4647955\gamma^6) > 0 \text{ for } \gamma \in [0.466, 1] \\
\tilde{\xi}_2 \equiv\ & 3125000\gamma^2\Gamma(0)^2(4175728 - 6082304\gamma + 1059524\gamma^2 + 3287264\gamma^3 - 2722052\gamma^4 \\
& + 822321\gamma^5 - 87219\gamma^6) > 0 \text{ for } \gamma \in [0.466, 1]
\end{aligned}$$

$$\tilde{\xi}_1 \equiv 156250000\gamma\Gamma(0)^3(3-\gamma)(17360-19144\gamma+868\gamma^2+8930\gamma^3-5308\gamma^4+861\gamma^5)$$
$$> 0 \text{ for } \gamma \in [0.466, 1]$$
$$\tilde{\xi}_0 \equiv 1953125000\Gamma(0)^4(3-\gamma)(336-352\gamma+40\gamma^2+128\gamma^3-83\gamma^4+14\gamma^5) > 0 \text{ for } \gamma \in [0.466, 1]$$

The derivative of welfare $\frac{dW_{uc}}{dk}$ is thus strictly positive for any k between $N-k^{**}$ and $\frac{78}{100}N$, welfare is strictly increasing with k and so we must have $k_{uc}^{opt} \geq \frac{78}{100}N$.

Having determined this lower bound on the size of the large CU, we now need to check that this size is such that the small union would be constrained and the large would not. If the large CU is larger than $\frac{78}{100}N$, then the small CU must be smaller than $\frac{22}{100}N$. As $\frac{22}{100}N < k^{**}(N,1)$, the small union is well constrained by Article XXIV.

We now need to check that the large union is larger than k^{**}. Recall that k^{**} is a monotonically decreasing function of γ and notice that $k^{**}(N,\frac{1}{2}) = \frac{3(N+1)}{4} < \frac{78}{100}N$ so for $\gamma \geq \frac{1}{2}$, by showing that $k_{uc}^{opt} \geq \frac{78}{100}N$, we have already shown that $k_{uc}^{opt} \geq k^{**}$. To finish the proof, we now need to show that $k_{uc}^{opt} \geq k^{**}$ for $\gamma \in (\hat{\gamma}(N), \frac{1}{2})$. Evaluating the first derivative of welfare (D.7) at k^{**} yields

$$\frac{dW_{uc}}{dk}(k^{**}, \{k^{**}, N-k^{**}\}) = -\frac{\delta_k(N,\gamma)}{2\Gamma(N)^2 D(1)^2} \tag{D.8}$$

From the study of (D.6) above, we know that $\delta_k(N,\gamma) < 0$ for $\gamma \in (\hat{\gamma}(N), \gamma_{\frac{N}{2}})$ and so from (D.8) we have that the derivative of welfare is strictly positive at k^{**}. Hence the welfare function is increasing for k between $N-k^{**}$ and k^{**} and so k_{uc}^{opt} must be greater than k^{**} and so the large CU cannot be constrained by Article XXIV.

3) Assume that no bloc is constrained by Article XXIV (UU):
As a final check, we assume that neither of the two blocs is constrained by Article XXIV and we show that this assumption leads to a contradiction.

Making use of (6), (7) and (8), but now with the large bloc imposing $\tau_c(k) = \tau(k)$ and the small bloc imposing $\tau_c(N-k) = \tau(N-k)$, we can again calculate the first derivative of the welfare function of the large bloc with respect to its size

$$\frac{dW_{uu}(k, \{k, N-k\})}{dk} = \frac{H_{uu}(k,N,\gamma)}{2D(k)^2 D(N-k)^3} \tag{D.9}$$

with

$$H_{uu}(k,N,\gamma) \equiv \tilde{\eta}_8(N,\gamma)k^8 + \tilde{\eta}_7(N,\gamma)k^7 + \tilde{\eta}_6(N,\gamma)k^6 + \tilde{\eta}_5(N,\gamma)k^5 + \tilde{\eta}_4(N,\gamma)k^4$$
$$+ \tilde{\eta}_3(N,\gamma)k^3 + \tilde{\eta}_2(N,\gamma)k^2 + \tilde{\eta}_1(N,\gamma)k + \tilde{\eta}_0$$

and

$$
\begin{aligned}
\tilde{\eta}_8(N,\gamma) \equiv\ & 32(1-\gamma)(2-\gamma)\gamma^8 \\
\tilde{\eta}_7(N,\gamma) \equiv\ & -16\gamma^7\left[\gamma(1-\gamma)(25-13\gamma)N+(2-\gamma)(8-17\gamma+5\gamma^2)\right] \\
\tilde{\eta}_6(N,\gamma) \equiv\ & 8\gamma^6\left[\gamma^2(1-\gamma)(142-95\gamma+16\gamma^2-3\gamma^3)N^2\right. \\
& +2\gamma(2-\gamma)(26-96\gamma+63\gamma^2-20\gamma^3+3\gamma^4)N \\
& \left.-(2-\gamma)^2(49-3\gamma-40\gamma^2+21\gamma^3-3\gamma^4)\right] \\
\tilde{\eta}_5(N,\gamma) \equiv\ & -4\gamma^5\left[\gamma^3(1-\gamma)(453-393\gamma+123\gamma^2-23\gamma^3)N^3\right. \\
& +\gamma^2(2-\gamma)(64-883\gamma+883\gamma^2-357\gamma^3+53\gamma^4)N^2 \\
& -\gamma(2-\gamma)^2(647-86\gamma-502\gamma^2+266\gamma^3-37\gamma^4)N \\
& \left.-(2-\gamma)^3(292-111\gamma-97\gamma^2+55\gamma^3-7\gamma^4)\right] \\
\tilde{\eta}_4(N,\gamma) \equiv\ & 2\gamma^4\left[2\gamma^4(1-\gamma)(415-439\gamma+177\gamma^2-33\gamma^3)N^4\right. \\
& -4\gamma^3(2-\gamma)(147+336\gamma-534\gamma^2+248\gamma^3-37\gamma^4)N^3 \\
& -\gamma^2(2-\gamma)^2(3905-2775\gamma-313\gamma^2+507\gamma^3-76\gamma^4)N^2 \\
& -2\gamma(2-\gamma)^3(1606-1679\gamma+656\gamma^2-133\gamma^3+14\gamma^4)N \\
& \left.-(2-\gamma)^4(682-1092\gamma+693\gamma^2-201\gamma^3+22\gamma^4)\right] \\
\tilde{\eta}_3(N,\gamma) \equiv\ & \gamma^3\left[-4\gamma^5(1-\gamma)(203-247\gamma+113\gamma^2-21\gamma^3)N^5\right. \\
& +4\gamma^4(2-\gamma)(688-411\gamma-177\gamma^2+167\gamma^3-27\gamma^4)N^4 \\
& +\gamma^3(2-\gamma)^2(12309-15884\gamma+7766\gamma^2-1916\gamma^3+221\gamma^4)N^3 \\
& +\gamma^2(2-\gamma)^3(14396-22735\gamma+14361\gamma^2-4253\gamma^3+487\gamma^4)N^2 \\
& +\gamma(2-\gamma)^4(7040-13540\gamma+9359\gamma^2-2774\gamma^3+299\gamma^4)N \\
& \left.+(2-\gamma)^5(1552-3056\gamma+2040\gamma^2-569\gamma^3+57\gamma^4)\right] \\
\tilde{\eta}_2(N,\gamma) \equiv\ & \gamma^2\left[4\gamma^6(1-\gamma)(41-55\gamma+27\gamma^2-5\gamma^3)N^6\right. \\
& -8\gamma^5(2-\gamma)(283-409\gamma+229\gamma^2-63\gamma^3+8\gamma^4)N^5 \\
& -\gamma^4(2-\gamma)^2(9023-15320\gamma+10214\gamma^2-3208\gamma^3+395\gamma^4)N^4 \\
& -2\gamma^3(2-\gamma)^3(5881-11367\gamma+8157\gamma^2-2557\gamma^3+294\gamma^4)N^3 \\
& -\gamma^2(2-\gamma)^4(6133-14034\gamma+10385\gamma^2-3134\gamma^3+334\gamma^4)N^2 \\
& -2\gamma(2-\gamma)^5(284-1065\gamma+830\gamma^2-235\gamma^3+22\gamma^4)N \\
& \left.+(2-\gamma)^6(3-\gamma)(174-204\gamma+87\gamma^2-13\gamma^3)\right]
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\eta}_1(N,\gamma) \equiv \gamma(2-\gamma)^2 \big[& 8g^6(41-55g+27g^2-5g^3)N^6 \\
& + \gamma^5(1915-3948\gamma+3022\gamma^2-1020\gamma^3+127\gamma^4)N^5 \\
& + \gamma^4(2-\gamma)(24-1791\gamma+1997\gamma^2-721\gamma^3+83\gamma^4)N^4 \\
& - 2\gamma^3(2-\gamma)^2(3094-4257\gamma+2360\gamma^2-634\gamma^3+69\gamma^4)N^3 \\
& - 2\gamma^2(2-\gamma)^3(4778-7414\gamma+4413\gamma^2-1187\gamma^3+121\gamma^4)N^2 \\
& - \gamma(2-\gamma)^4(3-\gamma)(2015-2435\gamma+985\gamma^2-133\gamma^3)N \\
& -(2-\gamma)^5(3-\gamma)^2(13-5\gamma)(12-5\gamma) \big] \\
\tilde{\eta}_0(N,\gamma) \equiv (2-\gamma)^2 \big[& \gamma^6(413-718\gamma+488\gamma^2-154\gamma^3+19\gamma^4)N^6 \\
& + 2\gamma^5(2-\gamma)(1251-2152\gamma+1424\gamma^2-428\gamma^3+49\gamma^4)N^5 \\
& + \gamma^4(2-\gamma)^2(6261-10536\gamma+6716\gamma^2-1914\gamma^3+205\gamma^4)N^4 \\
& + 4\gamma^3(2-\gamma)^3(2038-3313\gamma+2018\gamma^2-545\gamma^3+55\gamma^4)N^3 \\
& + \gamma^2(2-\gamma)^4(3-\gamma)(1883-2299\gamma+931\gamma^2-125\gamma^3)N^2 \\
& + 2\gamma(2-\gamma)^5(3-\gamma)^2(103-84\gamma+17\gamma^2)N \\
& +(2-\gamma)^6(3-\gamma)^3(7-3\gamma) \big]
\end{aligned}
$$

As in the (UC) case above, the optimal size of the large CU k_{uu}^{opt} is given by setting (D.9) equal to zero. It is again difficult to find a closed-form solution to this polynomial equation of degree 8 in k, but we can again provide a lower bound for k_{uu}^{opt} by studying further the derivative of the welfare function (D.9).

Tedious derivations (multiple successive differentiation as in the (UC) case above) show that the numerator of the derivative of welfare H_{uu} is a decreasing function of k for $k \in [\frac{N}{2}, \frac{9N}{10}]$. Furthermore, we have

$$
H_{uu}(\frac{88}{100}N, N, \gamma) = \frac{\hat{\Xi}(N,\gamma)}{152587890625} > 0 \text{ for } \gamma > 0.466
$$

where $\hat{\Xi}(N,\gamma) \equiv \hat{\xi}_8 N^8 + \hat{\xi}_7 N^7 + \hat{\xi}_6 N^6 + \hat{\xi}_5 N^5 + \hat{\xi}_4 N^4 + \hat{\xi}_3 N^3 + \hat{\xi}_2 N^2 + \hat{\xi}_1 N + \hat{\xi}_0$ with

$$
\begin{aligned}
\hat{\xi}_8 &\equiv 52272\gamma^8(1-\gamma)(-543+14509\gamma-18125\gamma^2+4375\gamma^3) \\
\hat{\xi}_7 &\equiv 118800\gamma^7(2-\gamma)(-13032+333843\gamma-575745\gamma^2+304625\gamma^3-49475\gamma^4) \\
\hat{\xi}_6 &\equiv 5625\gamma^6(2-\gamma)^2(-23581677+135143194\gamma-155087180\gamma^2+64307342\gamma^3-8887231\gamma^4) \\
\hat{\xi}_5 &\equiv 31250\gamma^5(2-\gamma)^3(-53486337+122581771\gamma-97377033\gamma^2+32422545\gamma^3-3856298\gamma^4) \\
\hat{\xi}_4 &\equiv 390625\gamma^4(2-\gamma)^4(17868441-36385396\gamma+26161984\gamma^2-8014138\gamma^3+892161\gamma^4) \\
\hat{\xi}_3 &\equiv 39062500\gamma^3(2-\gamma)^5(1408224-2486197\gamma+1600855\gamma^2-447928\gamma^3+46134\gamma^4) \\
\hat{\xi}_2 &\equiv 244140625\gamma^2(2-\gamma)^6(3-\gamma)(152841-196361\gamma+82233\gamma^2-11267\gamma^3)
\end{aligned}
$$

$$\hat{\xi}_1 \equiv 12207031250\gamma(2-\gamma)^7(3-\gamma)^2(859-725\gamma+150\gamma^2)$$
$$\hat{\xi}_0 \equiv 152587890625(2-\gamma)^8(3-\gamma)^3(7-3\gamma)$$

All the coefficients $\hat{\xi}_i$ are positive for $\gamma \in [0.466, 1]$ except $\hat{\xi}_5$ which changes signs and so straightforward successive differentiation with respect to N shows that $H_{uu}(\frac{88}{100}N, N, \gamma) > 0$ for $\gamma > 0.466$ and $N \geq 4$. Hence the welfare function W_{uu} is strictly increasing for $k \in [\frac{N}{2}, \frac{88N}{100}]$ and k^{opt}_{uu} must be greater or equal to $\frac{88}{100}N$. If the large CU is larger than $\frac{88}{100}N$, then the small CU must be smaller than $\frac{12}{100}N$. As $\frac{12}{100}N < k^{**}(N,1)$, this leads to a contradiction: a union smaller than $k^{**}(N,1)$ is necessarily constrained by Article XXIV. Therefore, in a two-bloc equilibrium, the small bloc is always constrained by Article XXIV. □

Proof of Proposition 5. We assume that the equilibrium CU structure consists of at most two blocs: a bloc of size k which forms first and a bloc of size $N-k$. From Lemma 7 we know that the two blocs will necessarily be asymmetric with the larger bloc forming first and so we have $k > N-k$. The aim of this proof is to determine how does the presence of the Article XXIV constraint affect the large CU's choice of its size.

1. Article XXIV binding on the large CU leads to a more asymmetric equilibrium CU structure: the goal here is to determine how a change in the CET of the large CU affects the large CU's willingness to accept more or less members, i.e. we want to determine the sign of
$$\frac{\partial}{\partial \tau_L}\frac{\partial W^L(k)}{\partial k}\bigg|_{\tau_L=\tau(1),k^{opt}_{cc}}$$
where τ_L is the external tariff imposed by the large union and W^L is the welfare of a member country of the large union. We are interested in the sign of this second derivative at the point where the large union is bound by Article XXIV $\tau_L = \tau(1)$ (because we want to see the local impact of removing Article XXIV and raising τ_L) and where the large union has chosen its size optimally k^{opt}_{cc} (case where both unions are constrained by Article XXIV). By the theorem of Schwarz (also known as Young's Theorem), we have $\frac{\partial}{\partial \tau_L}\frac{\partial W^L(k)}{dk} = \frac{\partial}{\partial k}\frac{\partial W^L(k)}{\partial \tau_L}$.

$$\frac{\partial W^L(k)}{\partial \tau_L} = \frac{N-k}{\Gamma(0)^2\Gamma(N)^2}\left[\Gamma(0)\Gamma(2k) - D(k)\tau_L\right]$$

and

$$\begin{aligned}\frac{\partial^2 W^L(k)}{\partial k \partial \tau_L} = \frac{1}{\Gamma(0)^2\Gamma(N)^2}\Bigg\{ & 6\gamma\tau_L k^2 \\ & + \left\{\tau_L\left[2\gamma\Gamma(0)\Gamma(N) - 6\gamma^2 N + 4\gamma\Gamma(0)\right] - 4\gamma\Gamma(0)\right\}k \\ & + \tau_L\left\{\Gamma(0)\Gamma(N)\left[\Gamma(0)-\gamma N\right] + \gamma^2N^2 + \Gamma(0)^2 - 2\gamma\Gamma(0)N\right\} + \Gamma(0)\left[\Gamma(2N) - 2\Gamma(0)\right]\Bigg\}\end{aligned} \tag{D.10}$$

Evaluating (D.10) at $\tau_L = \tau(1)$ and $k = k_{cc}^{opt}$ given by (D.5) yields

$$\frac{\partial}{\partial \tau_L}\frac{\partial W^L(k)}{\partial k}\bigg|_{\tau_L=\tau(1),k=k_{cc}^{opt}} = \frac{f^L(N,\gamma)}{2\Gamma(0)\Gamma(N)^2 D(1)\lambda_1(N,\gamma)^2}$$

with

$$\begin{aligned}
f^L(N,\gamma) &\equiv f_4^L(\gamma)N^4 + f_3^L(\gamma)N^3 + f_2^L(\gamma)N^2 + f_1^L(\gamma)N + f_0^L(\gamma)\\
f_4^L(\gamma) &\equiv -\gamma^4(2216 - 8532\gamma + 11422\gamma^2 - 5979\gamma^3 + 896\gamma^4 - 27\gamma^5)\\
f_3^L(\gamma) &\equiv -2\gamma^3(8896 - 38536\gamma + 63932\gamma^2 - 51294\gamma^3 + 19601\gamma^4 - 2552\gamma^5 + 73\gamma^6)\\
f_2^L(\gamma) &\equiv -8\gamma^2(6480 - 33072\gamma + 65152\gamma^2 - 66560\gamma^3 + 37847\gamma^4 - 11297\gamma^5 + 1293\gamma^6 - 35\gamma^7)\\
f_1^L(\gamma) &\equiv -2\gamma(34048 - 204608\gamma + 462048\gamma^2 - 562784\gamma^3 + 409136\gamma^4 - 180036\gamma^5 + 44102\gamma^6\\
&\quad - 4492\gamma^7 + 115\gamma^8)\\
f_0^L(\gamma) &\equiv -\Gamma(0)(19264 - 109696\gamma + 250320\gamma^2 - 298784\gamma^3 + 219532\gamma^4 - 98072\gamma^5 + 25663\gamma^6\\
&\quad - 2710\gamma^7 + 69\gamma^8)
\end{aligned}$$

From Proposition 4 we know that both CUs are bound only for $\gamma \in [0, \hat{\gamma}(N)]$. Given that $\hat{\gamma}(N) < 0.476$, for the remainder of the proof we will consider $\gamma \in [0, 0.476]$ which will cover the range of interest.

The coefficients $f_4^L(\gamma)$, $f_3^L(\gamma)$, $f_2^L(\gamma)$ and $f_0^L(\gamma)$ (of the fourth degree polynomial in N) are all negative for $\gamma \in [0, 0.476]$, but $f_1^L(\gamma)$ changes sign for $\gamma \in [0, 0.476]$. To sign $f^L(N,\gamma)$ we thus differentiate twice with respect to N.

$$\frac{d^2 f^L(N,\gamma)}{dN^2} = 12f_4^L(\gamma)N^2 + 6f_3^L(\gamma)N + 2f_2^L(\gamma) \leq 0 \text{ for } \gamma \in [0, 0.476]$$

and so the first derivative $\frac{df^L(N,\gamma)}{dN}$ is a decreasing function of N. Furthermore, $\frac{d}{dN}f^L(4,\gamma) = -2\gamma(34048 + 2752\gamma - 169248\gamma^2 - 44000\gamma^3 + 255856\gamma^4 + 30972\gamma^5 - 141866\gamma^6 + 29076\gamma^7 - 957\gamma^8) \leq 0$ for $\gamma \in [0, 0.476]$ and so $\frac{df^L(N,\gamma)}{dN}$ is negative for any $N \geq 4$ and $f^L(N,\gamma)$ is a decreasing function of N. Finally, $f^L(4,\gamma) = -38528 - 33728\gamma + 197088\gamma^2 + 246032\gamma^3 - 209720\gamma^4 - 336836\gamma^5 + 88074\gamma^6 + 145979\gamma^7 - 35136\gamma^8 + 1197\gamma^9 \leq 0$ for $\gamma \in [0, 0.476]$ and so we have

$$\frac{\partial}{\partial \tau_L}\frac{\partial W^L(k)}{\partial k}\bigg|_{\tau_L=\tau(1),k=k_{cc}^{opt}} \leq 0 \text{ for } \gamma \in [0, \hat{\gamma}(N)]$$

And so, when Article XXIV is binding on the big bloc, if it could raise its tariff, it would want to accept fewer members. □

2. Article XXIV binding on the small CU leads to a more symmetric equilibrium CU structure: the goal here is to determine how a change in the CET of the small CU affects the large CU's willingness to accept more or less members, i.e. we want to determine the sign of

$$\frac{\partial}{\partial \tau_S}\frac{\partial W^L(k)}{\partial k}\bigg|_{\tau_S=\tau(1),k=k^{opt}}$$

where τ_S is the external tariff imposed by the small union and W^L is the welfare of a member country of the large union. Again, we are interested in the sign of this second derivative at the point where the small union is constrained by Article XXIV $\tau_S = \tau(1)$ and where the large union has chosen its size optimally. Hence, when the large union is also constrained by Article XXIV ($\gamma \leq \hat{\gamma}(N)$), we want to evaluate the second derivative at $k = k^{opt}_{cc}$, and, when the large union is not constrained by Article XXIV ($\gamma > \hat{\gamma}(N)$), we want to evaluate the second derivative at $k = k^{opt}_{uc}$.

Differentiating (8) with respect to the tariff of the small union gives

$$\frac{\partial W^L(k)}{\partial \tau_S} = -\frac{2}{\Gamma(0)^2\Gamma(N)^2}\left[(N-k)\Gamma(N-k)\Gamma(0) - \tau_S(N-k)\Gamma(N-k)^2\right]$$

Differentiating with respect to k yields

$$\frac{\partial}{\partial k}\frac{\partial W^L(k)}{\partial \tau_S} = \frac{2}{\Gamma(0)^2\Gamma(N)^2}\left\{\Gamma(0)\Gamma[2(N-k)] - \tau_S\Gamma(N-k)\Gamma[3(N-k)]\right\} \qquad \text{(D.11)}$$

We want to evaluate this derivative at $\tau_S = \tau(1)$ and the optimal size of the large CU. The optimal size of the large CU depends on whether the large CU is constrained or not. We thus have to distinguish two cases:

2.a) For $\boldsymbol{\gamma \leq \hat{\gamma}(N) < 0.476}$: the optimal size is k^{opt}_{cc} given by (D.5). Evaluating (D.11) at $\tau_S = \tau(1)$ and $k = k^{opt}_{cc}$ yields

$$\frac{\partial}{\partial \tau_S}\frac{\partial W^L(k)}{\partial k}\bigg|_{\tau_S=\tau(1),k=k^{opt}_{cc}} = \frac{f^S(N,\gamma)}{2\Gamma(0)\Gamma(N)^2 D(1)\lambda_1(N,\gamma)^2}$$

with

$$\begin{aligned}
f^S(N,\gamma) &\equiv f^S_4(\gamma)N^4 + f^S_3(\gamma)N^3 + f^S_2(\gamma)N^2 + f^S_1(\gamma)N + f^S_0(\gamma)\\
f^S_4(\gamma) &\equiv \gamma^4(10 - 13\gamma + \gamma^2)(292 - 648\gamma + 373\gamma^2 - 15\gamma^3)\\
f^S_3(\gamma) &\equiv 16\gamma^3(1944 - 6852\gamma + 9414\gamma^2 - 6381\gamma^3 + 2070\gamma^4 - 194\gamma^5 + 5\gamma^6)\\
f^S_2(\gamma) &\equiv 2\gamma^2(64480 - 226224\gamma + 339520\gamma^2 - 283744\gamma^3 + 137498\gamma^4 - 35257\gamma^5 + 3078\gamma^6\\
&\qquad - 77\gamma^7)\\
f^S_1(\gamma) &\equiv 4\gamma(61952 - 212704\gamma + 339600\gamma^2 - 325104\gamma^3 + 198216\gamma^4 - 75878\gamma^5 + 16253\gamma^6\\
&\qquad - 1324\gamma^7 + 32\gamma^8)\\
f^S_0(\gamma) &\equiv (2-\gamma)(92992 - 258688\gamma + 382544\gamma^2 - 346784\gamma^3 + 209772\gamma^4 - 81208\gamma^5 + 18807\gamma^6\\
&\qquad - 1598\gamma^7 + 39\gamma^8)
\end{aligned}$$

$f^S(N,\gamma)$ is a fourth degree polynomial in N. All the coefficients $f_i^S(\gamma)$, $i = 0,\dots,4$, are positive for $\gamma \in [0, 0.476]$ and so $f^S(N,\gamma)$ is positive for any $N \geq 0$ and $\gamma \in [0, \hat{\gamma}(N)]$.

2.b) For $\boldsymbol{\gamma > \hat{\gamma}(N)} > \mathbf{0.466}$: we do not have a closed-form solution for the optimal size of the large union, but we know from the proof of Proposition 4 that $k_{uc}^{opt} \geq \frac{78}{100}N$. We will thus show that the second derivative (D.11) is positive for any $k \in [\frac{78}{100}N, N]$ and $\gamma \in [0.466, 1]$. Evaluating (D.11) at $\tau_S = \tau(1)$ yields

$$\frac{\partial}{\partial \tau_S}\frac{\partial W^L(k)}{\partial k}\bigg|_{\tau_S=\tau(1)} = \frac{2\tilde{f}(k,N,\gamma)}{\Gamma(0)\Gamma(N)^2 D(1)}$$

with

$$\begin{aligned}\tilde{f}(k,N,\gamma) \equiv & -3\gamma^2(2+\gamma)k^2 \\ & + \left[2\gamma(1+3\gamma)N - 4 + 8\gamma - 5\gamma^2\right]k \\ & + \gamma^2(2-9\gamma)N^2 + \gamma(16-26\gamma+13\gamma^2)N + 4(2-\gamma)(2-2\gamma+\gamma^2)\end{aligned}$$

$\tilde{f}(k,N,\gamma)$ is a second degree polynomial in k. The second derivative with respect to k is $\frac{d^2\tilde{f}(k,N,\gamma)}{dk^2} = -6\gamma^2(2+\gamma) < 0$ for $\gamma \in [0.466, 1]$. Hence the first derivative of $\tilde{f}$ is a decreasing function of k. The first derivative of $\tilde{f}$ evaluated at $k = \frac{78}{100}N$ can be either positive or negative. However, the first derivative evaluated at $k = N$ is strictly negative for $\gamma \in [0.466, 1]$

$$\frac{d\tilde{f}}{dk}(N,N,\gamma) = -2\gamma^2(4-3\gamma)N - 2\gamma(4-8\gamma+5\gamma^2) < 0$$

And so $\tilde{f}$ is either a monotonically decreasing function of k or it is initially an increasing function of k and then a decreasing function of k. In either case, if we prove that $\tilde{f}$ is positive at both bounds of the considered interval, we will have proven that it is positive on the entire interval. Evaluating $\tilde{f}$ at the lower bound $k = \frac{78}{100}N$ yields

$$\begin{aligned}\tilde{f}(\frac{78}{100}N,N,\gamma) = \frac{1}{2500}\Bigg[& \underbrace{11\gamma^2(334-333\gamma)}_{>0 \text{ for } \gamma\in[0.466,1]}N^2 \\ & + \underbrace{200\gamma(122-169\gamma+65\gamma^2)}_{>0 \text{ for } \gamma\in[0.466,1]}N \\ & + \underbrace{10000(2-\gamma)(2-2\gamma+\gamma^2)}_{>0 \text{ for } \gamma\in[0.466,1]}\Bigg] > 0\end{aligned}$$

Evaluating $\tilde{f}$ at the upper bound $k = N$ yields

$$\tilde{f}(N, N, \gamma) = \gamma(4 - 3\gamma)N + 4(2 - 2\gamma + \gamma^2) > 0$$

And so $\tilde{f}$ is positive for any $k \in [\frac{78}{100}N, N]$ and $\gamma \in [0.466, 1]$, and so it must be positive at $k = k_{uc}^{opt}$. And so, when Article XXIV is binding on the small bloc, if the small bloc could raise its tariff, the large bloc would want to accept more members. $\square$

Proof of Proposition 6. As explained in the the proof of Proposition 4, for $\gamma > \hat{\gamma}(N)$, it is difficult to find a closed-form for the optimum size of the large CU both with and without Article XXIV. However, by studying the derivative of welfare of the large CU with respect to its size, we are able to determine an upper bound for the size of the large CU with Article XXIV: $k_{uc}^{opt} < \frac{8}{9}N$, and a lower bound for the size of the large CU without Article XXIV: $k_{uu}^{opt} > \frac{8}{9}N + 1$, which shows that the CU structure with Article XXIV is strictly more symmetric. The proof proceeds in two steps: first, we determine an upper bound for k_{uc}^{opt}, and second, we determine a lower bound for k_{uu}^{opt}.

1) We show that $\boldsymbol{k_{uc}^{opt} < \frac{8}{9}N}$: From the proof of Proposition 4 we know that the numerator of the derivative of welfare of the large CU with respect to its size, $H_{uc}(k, N, \gamma)$, changes sign only once in the interval $[N - k^{**}, N]$. It is initially positive and increasing function and then it becomes a decreasing function and negative. We show that $H_{uc}(\frac{8N}{9}, N, \gamma) < 0$ and so $k_{uc}^{opt} < \frac{8}{9}N$. Evaluating $H_{uc}(k, N, \gamma)$ at $k = \frac{8}{9}N$ yields

$$H_{uc}(\frac{8N}{9}, N, \gamma) = \frac{H_{uc}^{8/9}(N, \gamma)}{177147}$$

with

$$\begin{aligned}
H_{uc}^{8/9}(N, \gamma) \equiv{}& h_6(\gamma)N^6 + h_5(\gamma)N^5 + h_4(\gamma)N^4 + h_3(\gamma)N^3 + h_2(\gamma)N^2 + h_1(\gamma)N + h_0(\gamma) \\
h_6(\gamma) \equiv{}& -128\gamma^6(1 - \gamma)(107524 - 135544\gamma + 42003\gamma^2 - 2997\gamma^3) \\
h_5(\gamma) \equiv{}& -96\gamma^5(1170608 - 3326704\gamma + 3733920\gamma^2 - 2008600\gamma^3 + 481893\gamma^4 - 39663\gamma^5) \\
h_4(\gamma) \equiv{}& -27\gamma^4(6770112 - 34686816\gamma + 65461812\gamma^2 - 59159264\gamma^3 + 27345557\gamma^4 \\
& - 6098893\gamma^5 + 511422\gamma^6) \\
h_3(\gamma) \equiv{}& 972\gamma^3(2 - \gamma)(444816 - 150388\gamma - 1328708\gamma^2 + 1891005\gamma^3 - 1072878\gamma^4 \\
& + 273551\gamma^5 - 25610\gamma^6) \\
h_2(\gamma) \equiv{}& 4374\gamma^2(2 - \gamma)^2(221288 - 283896\gamma - 44978\gamma^2 + 273484\gamma^3 - 192581\gamma^4 \\
& + 54631\gamma^5 - 5522\gamma^6) \\
h_1(\gamma) \equiv{}& 78732\gamma(2 - \gamma)^3(3 - \gamma)(2968 - 3148\gamma - 126\gamma^2 + 1725\gamma^3 - 977\gamma^4 + 154\gamma^5) \\
h_0(\gamma) \equiv{}& 177147(2 - \gamma)^4(3 - \gamma)(336 - 352\gamma + 40\gamma^2 + 128\gamma^3 - 83\gamma^4 + 14\gamma^5)
\end{aligned}$$

$H_{uc}^{8/9}(N,\gamma)$ is a sixth degree polynomial in N. We determine its sign by differentiating it successively. The coefficients $h_6(\gamma)$ and $h_5(\gamma)$ are both negative for $\gamma \in [0.466, 1]$ and so the fifth derivative of $H_{uc}^{8/9}(N,\gamma)$ with respect to N is negative for $\gamma \in [\hat{\gamma}(N), 1]$. Therefore, the fourth derivative is decreasing with N. Evaluating the fourth derivative at $N = 4$ yields

$$\begin{aligned}\frac{d^4}{dN^4}H_{uc}^{8/9}(4,\gamma) = &-72\gamma^4(60931008 + 437007776\gamma - 438888492\gamma^2 - 631740896\gamma^3 \\ &+ 778687293\gamma^4 - 207278517\gamma^5 + 9907758\gamma^6) < 0 \text{ for } \gamma \in [0.466, 1]\end{aligned}$$

Hence the fourth derivative is negative for any $N \geq 4$ and the third derivative is decreasing. Evaluating the third derivative at $N = 4$ yields

$$\begin{aligned}\frac{d^3}{dN^3}H_{uc}^{8/9}(4,\gamma) = &-24\gamma^3(-216180576 + 912350952\gamma + 1358166396\gamma^2 - 2542389098\gamma^3 \\ &- 1026080069\gamma^4 + 2118966136\gamma^5 - 572491971\gamma^6 + 19461546\gamma^7) \\ &< 0 \text{ for } \gamma \in [0.466, 1]\end{aligned}$$

Hence the third derivative is negative for any $N \geq 4$ and the second derivative is decreasing. Evaluating the second derivative at $N = 5$ yields

$$\begin{aligned}\frac{d^2}{dN^2}H_{uc}^{8/9}(5,\gamma) = &-24\gamma^2(-322637904 - 344344608\gamma + 2761805160\gamma^2 + 2684481116\gamma^3 \\ &- 5934883569\gamma^4 - 2489306999\gamma^5 + 5087448636\gamma^6 - 1372813698\gamma^7 \\ &+ 46571544\gamma^8) < 0 \text{ for } \gamma \in [0.466, 1]\end{aligned}$$

Hence the second derivative is negative for any $N \geq 5$ and the first derivative is decreasing. Evaluating the first derivative at $N = 7$ yields

$$\begin{aligned}\frac{d}{dN}H_{uc}^{8/9}(7,\gamma) = &-48\gamma(-116838288 - 791104968\gamma - 438540156\gamma^2 + 6064487874\gamma^3 \\ &+ 7654428044\gamma^4 - 13375288074\gamma^5 - 10877774936\gamma^6 + 16327506879\gamma^7 \\ &- 4364846022\gamma^8 + 178868088\gamma^9) < 0 \text{ for } \gamma \in [0.466, 1]\end{aligned}$$

Hence the first derivative is negative for any $N \geq 7$ and $H_{uc}^{8/9}(N,\gamma)$ is decreasing in N for $N \geq 7$. Evaluating $H_{uc}^{8/9}(N,\gamma)$ at $N = 9$ yields

$$\begin{aligned}H_{uc}^{8/9}(9,\gamma) = &-1417176(1 + 4\gamma)(-2016 - 20736\gamma - 44808\gamma^2 + 133720\gamma^3 + 426240\gamma^4 \\ &- 318584\gamma^5 - 812917\gamma^6 + 852991\gamma^7 - 214960\gamma^8 + 9408\gamma^9) < 0 \text{ for } \gamma \in [0.466, 1]\end{aligned}$$

And so $H_{uc}(\frac{8N}{9}, N, \gamma)$ is negative for any $N \geq 9$ and $\gamma \in [0.466, 1]$. Thus, for $N \geq 9$, we know that the derivative of welfare of the large CU with respect to its size is strictly negative for any $k \in [\frac{8N}{9}, N]$ and so the welfare function is decreasing on this interval and hence we must have $k_{uc}^{opt} < \frac{8N}{9}$.

2) We show that $\boldsymbol{k_{uu}^{opt} > \frac{8N}{9} + 1}$: From the proof of Proposition 4 we know that the numerator of the derivative of welfare H_{uu} is a decreasing function of k for $k \in [\frac{N}{2}, \frac{9N}{10}]$. Furthermore, we have

$$H_{uu}(\frac{8N}{9} + 1, N, \gamma) = \frac{H_{uu}^{8/9N+1}(N,\gamma)}{43046721}$$

with

$$\begin{aligned} H_{uu}^{8/9N+1}(N,\gamma) \equiv\, & \tilde{h}_8(\gamma)N^8 + \tilde{h}_7(\gamma)N^7 + \tilde{h}_6(\gamma)N^6 + \tilde{h}_5(\gamma)N^5 + \tilde{h}_4(\gamma)N^4 + \tilde{h}_3(\gamma)N^3 \\ & + \tilde{h}_2(\gamma)N^2 + \tilde{h}_1(\gamma)N + \tilde{h}_0(\gamma) \end{aligned}$$

and

$$\begin{aligned}
\tilde{h}_8(\gamma) \equiv\, & 256\gamma^8(1-\gamma)(-191 + 1009\gamma - 1053\gamma^2 + 243\gamma^3) \\
\tilde{h}_7(\gamma) \equiv\, & 576\gamma^7(-9932 + 54877\gamma - 95199\gamma^2 + 73923\gamma^3 - 27441\gamma^4 + 3780\gamma^5) \\
\tilde{h}_6(\gamma) \equiv\, & 81\gamma^6(-4268396 + 14080084\gamma - 16960511\gamma^2 + 9412786\gamma^3 - 2040132\gamma^4 \\
& - 93198\gamma^5 + 72231\gamma^6) \\
\tilde{h}_5(\gamma) \equiv\, & -729\gamma^5(10223696 - 9286212\gamma - 39303416\gamma^2 + 94486143\gamma^3 - 90787556\gamma^4 \\
& + 45666110\gamma^5 - 11899684\gamma^6 + 1264375\gamma^7) \\
\tilde{h}_4(\gamma) \equiv\, & 6561\gamma^4(798544 - 65351776\gamma + 250517124\gamma^2 - 427084256\gamma^3 + 411981428\gamma^4 \\
& - 241320219\gamma^5 + 85459655\gamma^6 - 16879111\gamma^7 + 1428999\gamma^8) \\
\tilde{h}_3(\gamma) \equiv\, & -59049\gamma^3(-6975232 + 79321024\gamma - 268606032\gamma^2 + 461989604\gamma^3 - 474534568\gamma^4 \\
& + 308796037\gamma^5 - 128155312\gamma^6 + 32625848\gamma^7 - 4574638\gamma^8 + 264497\gamma^9) \\
\tilde{h}_2(\gamma) \equiv\, & 531441\gamma^2(3567168 - 34340928\gamma + 116308704\gamma^2 - 208912384\gamma^3 + 229568260\gamma^4 \\
& - 163316936\gamma^5 + 76134788\gamma^6 - 22723839\gamma^7 + 4063152\gamma^8 - 375877\gamma^9 + 12062\gamma^{10}) \\
\tilde{h}_1(\gamma) \equiv\, & 4782969\gamma(698112 - 6446016\gamma + 22385280\gamma^2 - 42176912\gamma^3 + 49261712\gamma^4 \\
& - 37651524\gamma^5 + 19073408\gamma^6 - 6276883\gamma^7 + 1263162\gamma^8 - 134475\gamma^9 + 4364\gamma^{10} + 228\gamma^{11}) \\
\tilde{h}_0(\gamma) \equiv\, & 43046721(2-\gamma)(24192 - 209088\gamma + 684576\gamma^2 - 1203280\gamma^3 + 1280584\gamma^4 \\
& - 854948\gamma^5 + 350470\gamma^6 - 79263\gamma^7 + 6007\gamma^8 + 1090\gamma^9 - 236\gamma^{10} + 8\gamma^{11})
\end{aligned}$$

$H_{uu}^{8/9N+1}(N,\gamma)$ is an eighth degree polynomial in N which we sign again by successive differentiation. We have $\tilde{h}_8(\gamma) \geq 0$ and $\tilde{h}_7(\gamma) > 0$ for $\gamma \in [0.466, 1]$. The seventh derivative of $H_{uu}^{8/9N+1}(N,\gamma)$ with respect to N is thus strictly positive and the sixth derivative is increasing. Evaluating the sixth derivative at $N = 14$ yields

$$\frac{d^6}{dN^6} H_{uu}^{8/9N+1}(14, \gamma) = 720\gamma^6(-345740076 + 579845268\gamma + 1455554257\gamma^2 - 2925443886\gamma^3 + 1110593276\gamma^4 + 264248082\gamma^5 - 122173353\gamma^6) > 0 \text{ for } \gamma \in [0.466, 1]$$

Hence the sixth derivative is positive for any $N \geq 14$ and the fifth derivative is increasing. Evaluating the fifth derivative at $N = 37$ yields

$$\frac{d^5}{dN^5} H_{uu}^{8/9N+1}(37, \gamma) = 120\gamma^5(-7453074384 - 69984648324\gamma + 117372061584\gamma^2 + 396170374871\gamma^3 - 469602069600\gamma^4 - 343198770758\gamma^5 + 493697013984\gamma^6 - 113485470957\gamma^7) > 0 \text{ for } \gamma \in [0.466, 1]$$

Hence the fifth derivative is positive for any $N \geq 37$ and the fourth derivative is increasing. Evaluating the fourth derivative at $N = 57$ yields

$$\frac{d^4}{dN^4} H_{uu}^{8/9N+1}(57, \gamma) = 1944\gamma^4(64682064 - 31517274096\gamma - 163909258236\gamma^2 + 294621819684\gamma^3 + 1047807767128\gamma^4 - 913479950409\gamma^5 - 1617839257915\gamma^6 + 1786406339979\gamma^7 - 392868234531\gamma^8) > 0 \text{ for } \gamma \in [0.466, 1]$$

Hence the fourth derivative is positive for any $N \geq 57$ and the third derivative is increasing. Evaluating the third derivative at $N = 78$ yields

$$\frac{d^3}{dN^3} H_{uu}^{8/9N+1}(78, \gamma) = 1458\gamma^3(1694981376 - 12548074176\gamma - 2351281089168\gamma^2 - 9810826864236\gamma^3 + 17736126317880\gamma^4 + 68479566758905\gamma^5 - 43599047912520\gamma^6 - 138292111396576\gamma^7 + 138217523336946\gamma^8 - 29772894518163\gamma^9) > 0 \text{ for } \gamma \in [0.466, 1]$$

Hence the third derivative is positive for any $N \geq 78$ and the second derivative is increasing. Evaluating the second derivative at $N = 99$ yields

$$\frac{d^2}{dN^2} H_{uu}^{8/9N+1}(99, \gamma) = 1062882\gamma^2(3567168 + 195841728\gamma - 1921542144\gamma^2 - 174867696464\gamma^3 - 646944962468\gamma^4 + 1170952601564\gamma^5 + 4759434546420\gamma^6 - 2263227348443\gamma^7 - 11178475532846\gamma^8 + 10643459033365\gamma^9 - 2267493934194\gamma^{10}) > 0 \text{ for } \gamma \in [0.466, 1]$$

Hence the second derivative is positive for any $N \geq 99$ and the first derivative is increasing. Evaluating the first derivative at $N = 119$ yields

$$\begin{aligned}\frac{d}{dN} H_{uu}^{8/9N+1}(119, \gamma) = \gamma(&3339048054528 + 420354865621440\gamma + 13261444138967040\gamma^2 \\ &- 149157832409171280\gamma^3 - 971554276136207361 6\gamma^4 \\ &- 3276664761739050644 4\gamma^5 + 6058631530985489949 6\gamma^6 \\ &+ 2478371500469522048 29\gamma^7 - 9434653447057206838 8\gamma^8 \\ &- 6340067810212311527 41\gamma^9 + 5896255550356447067 88\gamma^{10} \\ &- 1249196233516412302 44\gamma^{11}) > 0 \text{ for } \gamma \in [0.466, 1]\end{aligned}$$

Hence the first derivative is positive for any $N \geq 119$ and $H_{uu}^{8/9N+1}(N, \gamma)$ is increasing in N for $N \geq 119$. Evaluating $H_{uu}^{8/9N+1}(N, \gamma)$ at $N = 139$ yields

$$\begin{aligned}H_{uu}^{8/9N+1}(139, \gamma) = &2082772548864 + 445085187704064\gamma + 32409995470751232\gamma^2 \\ &+ 768291758281588608\gamma^3 - 9456782826434012640\gamma^4 \\ &- 50630893009778971411 2\gamma^5 - 15997560686763089529 60\gamma^6 \\ &+ 30047206974571363021 20\gamma^7 + 12353849957872691636 641\gamma^8 \\ &- 38023315352566289655 45\gamma^9 - 33588024046516040228 650\gamma^{10} \\ &+ 30743031928745898281 124\gamma^{11} - 64882639338212232971 28\gamma^{12} \\ &> 0 \text{ for } \gamma \in [0.466, 1]\end{aligned}$$

And so $H_{uu}(\frac{8N}{9}, N, \gamma)$ is positive for any $N \geq 139$ and $\gamma \in [0.466, 1]$. Thus, for $N \geq 139$, we know that the derivative of welfare of the large CU with respect to its size is strictly positive for any $k \in [\frac{N}{2}, \frac{8N}{9} + 1]$ and so the welfare function is increasing on this interval and hence we must have $k_{uu}^{opt} > \frac{8N}{9} + 1$.

Thus we have for $N \geq 139$ and $\gamma \in [\hat{\gamma}(N), 1]$, $k_{uc}^{opt} < \frac{8N}{9} < \frac{8N}{9} + 1 < k_{uu}^{opt}$. $\square$

E CU formation algorithm

To determine the equilibrium CU structure (the number of CUs and their size), we numerically solve backwards the bloc formation game. To do so, for every N and γ, we run a grid search over the possible partitions of the N countries. We make use of the two following results derived in Lemmas 5 and 7: 1) the equilibrium CU structure is asymmetric; 2) there are at most four CUs in equilibrium. These two results allow us to significantly restrict the number of partitions we need to consider.

The calculation algorithm is as follows: Take a given N and γ. Assume that there will be at most four CUs in equilibrium. Let us denote these four blocs in order of formation 1, 2, 3 and 4, and their respective size k_1, k_2, k_3 and k_4 knowing that $k_1 \geq k_2 \geq k_3 \geq k_4$. (Note that we are not imposing that there will be four CUs, but at most four CUs. Some of these CUs may be empty).

Once CUs 1 and 2 form, the third CU chooses its size k_3 to maximize its welfare W_3. The fourth union is formed by the remaining countries $k_4 = N - (k_1 + k_2 + k_3)$. Thus, for a given k_1 and k_2, the third union solves

$$\underset{k_3}{\operatorname{argmax}}\, W_3(k_3, \{k_1, k_2, k_3, N - (k_1 + k_2 + k_3)\})$$

Hence for any $k_1 \in [\frac{N}{4}, N]$ and for any $k_2 \in [\frac{N-k_1}{3}, N-k_1]$, we need to find $k_3 \in [\frac{N-k_1-k_2}{2}, N-k_1-k_2]$ which maximizes welfare of CU 3. This gives us a "reaction function" $k_3^*(k_1, k_2)$.

Then we move on to calculate the size of the second union. Again, once CU 1 forms, the second union chooses its size k_2 to maximize its welfare W_2 knowing $k_3^*(k_1, k_2)$. So for a given k_1, the second union solves

$$\underset{k_2}{\operatorname{argmax}}\, W_2(k_2, \{k_1, k_2, k_3^*(k_1, k_2), N - (k_1 + k_2 + k_3^*(k_1, k_2)\})$$

Hence for any $k_1 \in [\frac{N}{4}, N]$, we need to find $k_2 \in [\frac{N-k_1}{3}, N - k_1]$ which maximizes welfare of CU 2 knowing $k_3^*(k_1, k_2)$. This gives us another reaction function $k_2^*(k_1)$. Finally, we determine the size of the first CU which chooses k_1 to maximize its welfare W_1 knowing $k_2^*(k_1)$ and $k_3^*(k_1, k_2^*(k_1))$. We solve

$$\underset{k_1}{\operatorname{argmax}}\, W_1(k_1, \{k_1, k_2^*(k_1), k_3^*(k_1, k_2^*(k_1)), N - (k_1 + k_2^*(k_1) + k_3^*(k_1, k_2^*(k_1))\})$$

We run this grid search for $0 \leq \gamma \leq 1$ (varying γ by $2 \cdot 10^{-6}$) and for $N = 4, \ldots, 10^5$.

References

Yi, S.S., 1996. Endogenous formation of customs unions under imperfect competition: Open regionalism is good. Journal of International Economics 41, 153–177.