Not so Demanding: Demand Structure and Firm Behavior

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Supplementary Online Appendix B

B1. Preliminaries: A Key Lemma

We make repeated use of the following result:

LEMMA 4: *Consider a twice-differentiable function $g(x)$. Both the double-log convexity of $g(x)$ and the rate of change of its elasticity can be expressed in terms of its first and second derivatives as follows:*

$$\frac{d^2 \log g}{d(\log x)^2} = x\frac{d}{dx}\left(\frac{xg'}{g}\right) = \frac{xg'}{g}\left(1 - \frac{xg'}{g} + \frac{xg''}{g'}\right) \tag{B1}$$

For most of the paper, $g(x)$ is the inverse demand function $p(x)$, and (B1) can be expressed in terms of the demand elasticity and convexity:

$$\frac{d^2 \log p}{d(\log x)^2} = \frac{x\varepsilon_x}{\varepsilon^2} = -\frac{1}{\varepsilon}\left(1 + \frac{1}{\varepsilon} - \rho\right) \tag{B2}$$

Recalling equation (4), this gives the result in Section I.B that the elasticity of demand increases with sales if and only demand is superconvex. Qualitatively the same outcome comes from applying Lemma 4 to the direct demand function, replacing $g(x)$ by $x(p)$, and making use of (A1) and (A2):

$$\frac{d^2 \log x}{d(\log p)^2} = -p\frac{d\varepsilon}{dp} = -\varepsilon\left(1 + \varepsilon - \varepsilon\rho\right) \tag{B3}$$

We use a different application of the Lemma to prove Proposition 1 in Appendix B2 below. Now, let $g(x)$ denote the absolute value of the demand slope $-p'(x)$, so (B1) becomes:

$$\frac{d^2 \log(-p')}{d(\log x)^2} = -x\rho_x = -\rho\left(1 + \rho - \chi\right) \tag{B4}$$

where χ is the temperance parameter, defined in Appendix A1. The result in (B4) that the change in convexity as sales rise depends only on temperance and convexity itself parallels that in (B2) that the change in elasticity as sales rise depends only on convexity and elasticity itself.

All these expressions are zero in the CES case given by (4), when all three parameters depend only on the elasticity σ: $\{\varepsilon, \rho, \chi\}_{CES} = \left\{\sigma, 1 + \frac{1}{\sigma}, 2 + \frac{1}{\sigma}\right\}$.

B2. Proof of Proposition 1: Existence of the Manifold

We wish to prove that, except in the CES case, only one of ε_x and ρ_x can be zero at any x. Recall from equations (B2) and (B4) that $\varepsilon_x = \frac{\varepsilon}{x}\left(\rho - \frac{\varepsilon+1}{\varepsilon}\right)$ and $\rho_x = \frac{\rho}{x}(1+\rho-\chi)$, where $\chi \equiv -\frac{xp'''}{p''}$. We have already seen that ε_x can be zero only along the CES locus. As for $\rho_x = 0$, there are two cases where it can equal zero. The first is where $\rho = 0$. From (B4), this implies that ε_x equals $-\frac{\varepsilon+1}{x}$, which is non-zero. The second is where $1+\rho-\chi = 0$. As we saw in Section II.D, this implies that the demand function takes the iso-convex or Bulow-Pfleiderer form: $p(x) = \alpha + \beta x^{-\theta}$. The intersection of this with $\varepsilon_x = 0$ is the CES limiting case of Bulow-Pfleiderer as sales tend towards zero: see Figure B3(a) below. Hence we can conclude that the only cases where both ε_x and ρ_x equal zero at a given x lie on a CES demand function.

B3. Proof of Proposition 2: Manifold Invariance

We wish to prove that one or other of the conditions in Proposition 2 is necessary and sufficient for a given demand manifold to be invariant with respect to a demand parameter ϕ. Of the two conditions, one relates to properties of inverse demand functions and the other to those of direct demand functions. However, it is clear by inspection that the two conditions are equivalent to each other except that ε and ρ are functions of x in one case but of p in the other. Hence we need only prove the conditions in one case. In what follows, we consider only the case where ε and ρ are derived from the inverse demand function and so depend on x and ϕ. The restriction we derive is thus necessary and sufficient for manifold invariance, conditional on the elasticity and convexity being derived from the inverse demand function.[1]

Recall that, when ρ_x is non-zero, which implies that ε_x is also non-zero, the demand manifold can be locally written either as $\varepsilon = \bar{\varepsilon}(\rho, \phi)$ or as $\rho = \bar{\rho}(\varepsilon, \phi)$ where ϕ is a vector parameter.[2]

Let us first prove sufficiency. Suppose ε and ρ depend on x and ϕ through a common sub-function of x and ϕ as stated in the proposition: $\varepsilon(x,\phi) = \tilde{\varepsilon}[F(x,\phi)]$ and $\rho(x,\phi) = \tilde{\rho}[F(x,\phi)]$. Eliminating F from this system yields a relation between ε and ρ, which is independent of ϕ.

Now let us prove necessity. Without loss of generality, suppose that the demand manifold can be locally written as:

$$\varepsilon = \bar{\varepsilon}(\rho, \phi) \tag{B5}$$

[1] We are extremely grateful to Ernst Hairer for this proof, which replaces our earlier feeble attempts.

[2] As explained in the text, we exclude from the proposition all Bulow-Pfleiderer demands, for which ρ_x is zero, and in particular the CES special case, for which ε_x is also zero. This also excludes the case where ρ_p is zero. Since $\rho(p) = \frac{x(p)x''(p)}{(x'(p))^2}$ from (A2), the only demand function with $\rho_p = 0$ is the linear, which is a special case of the Bulow-Pfleiderer class.

Let us define F by $F(x,\phi)=\rho(x,\phi)$. From (B5), we have $\varepsilon(x,\phi)=\bar{\varepsilon}[\rho(x,\phi),\phi]$. Since the demand manifold is independent of ϕ by assumption, we have $\bar{\varepsilon}(\rho,\phi)=\bar{\varepsilon}(\rho)$ and so $\varepsilon(x,\phi)=\bar{\varepsilon}[\rho(x,\phi)]=\bar{\varepsilon}[F(x,\phi)]$. Hence both ε and ρ depend on x and ϕ only through the common sub-function F.

B4. Proof of Corollary 2: Multiplicative Separability

If demands are multiplicatively separable in ϕ, both the elasticity and convexity are independent of ϕ. In the case of inverse demands, $p(x,\phi)=\beta(\phi)\tilde{p}(x)$ implies:

$$\text{(B6)}\qquad \varepsilon=-\frac{p(x,\phi)}{xp_x(x,\phi)}=-\frac{\tilde{p}(x)}{x\tilde{p}'(x)}\quad\text{and}\quad\rho=-\frac{xp_{xx}(x,\phi)}{p_x(x,\phi)}=-\frac{x\tilde{p}''(x)}{\tilde{p}'(x)}$$

A special case of this is additive preferences: $\int_{\omega\in\Omega}u\left[x(\omega)\right]d\omega$. The first-order condition is $u'\left[x(\omega)\right]=\lambda^{-1}p(\omega)$, which implies that the perceived indirect demand function can be written in multiplicative form: $p(x,\phi)=\lambda(\phi)\tilde{p}(x)$.

Similar derivations hold for direct demands. If $x(p,\phi)=\delta(\phi)\,\tilde{x}(p)$ then:

$$\text{(B7)}\qquad \varepsilon=-\frac{px_p(p,\phi)}{x(p,\phi)}=-\frac{p\tilde{x}'(p)}{\tilde{x}(p)}\quad\text{and}\quad\rho=\frac{x(p,\phi)x_{pp}(p,\phi)}{[x_p(p,\phi)]^2}=\frac{\tilde{x}(p)\tilde{x}''(p)}{[\tilde{x}'(p)]^2}$$

We also have a similar corollary, the case of indirect additivity, where the indirect utility function can be written as: $\int_{\omega\in\Omega}v\left[p(\omega)/I\right]d\omega$. Roy's Identity implies that: $v'\left[p(\omega)/I\right]=-\lambda x(\omega)$, where λ is the marginal utility of income, from which the direct demand function facing a firm can be written in multiplicative form: $x(p/I,\phi)=-\lambda^{-1}(\phi)\tilde{x}(p/I)$.

B5. Market Size and the Logistic Demand Function

To illustrate Corollary 4 that the demand manifold is independent of market size, consider the logistic direct demand function, equivalent to a logit inverse demand function (see Cowan (2012)):

$$\text{(B8)}\qquad x(p,s)=\left(1+e^{p-a}\right)^{-1}s\quad\Leftrightarrow\quad p(x,s)=a-\log\frac{x}{s-x}$$

Here x/s is the share of the market served: $x\in[0,s]$; and a is the price that induces a 50% market share: $p=a$ implies $x=\frac{s}{2}$. The elasticity equals $\varepsilon=p\frac{s-x}{s}$, while the convexity equals $\rho=\frac{s-2x}{s-x}$, which must be less than one. Eliminating x and p yields a closed-form expression for the manifold:

$$\text{(B9)}\qquad \bar{\varepsilon}(\rho)=\frac{a-\log(1-\rho)}{2-\rho}$$

which is invariant with respect to market size s though not with respect to a. Figure B1 illustrates this for values of a equal to 2 and 5.[3]

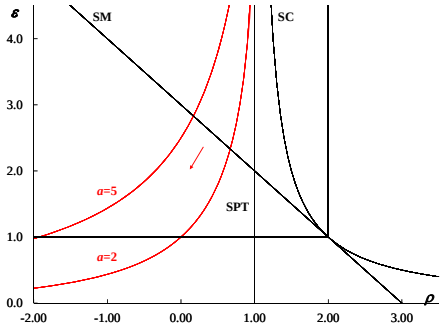


FIGURE B1. THE DEMAND MANIFOLD FOR THE LOGISTIC DEMAND FUNCTION

The logistic is just one example of a whole family of demand functions, many of which can be derived from log-concave distribution functions: Bergstrom and Bagnoli (2005) give a comprehensive review of these. The power of the approach introduced in the last section is that we can immediately state the properties of all these functions: they imply sub-pass-through and, *a fortiori*, subconvexity, while they are typically supermodular for low values of output and submodular for high values. Any shock, such as a partial-equilibrium increase in market size, that raises the output of a monopoly firm, implies an adjustment as shown by the arrow in the figure.

B6. Proof of Proposition 3: Bipower Demands

The inverse and direct bipower demand functions in (13a) and (13b) have very different implications for behavior. However, they have the same functional form except that the roles of x and p are reversed, so results proved for one can be applied immediately to the other. It is most convenient to focus on the inverse demands in (13a). Sufficiency follows by differentiating $p(x)$ and calculating the manifold directly. As in Section A3, we write the elasticity and convexity in a way which shows that they satisfy the conditions for manifold invariance in condition (11a) from Proposition 2. Necessity follows by setting $\rho(x) = a + b\varepsilon(x)$, where a and b are constants, and solving the resulting Euler-Cauchy differential equation. This proves the result in (13a): a bipower inverse demand function is necessary and sufficient for a manifold such that ρ is affine in ε. With appropriate

[3]The value of ρ determines market share and the level of price relative to a: $x = \frac{1-\rho}{2-\rho}s$ and $p = a - \log(1-\rho)$. In particular, when the function switches from convex to concave (i.e., ρ is zero), the elasticity equals $\frac{a}{2}$, market share is 50%, and $p = a$.

relabeling this in turn implies that a bipower direct demand function is necessary and sufficient for an affine *dual* manifold, that is to say, an equation linking the dual parameters r and e: $\bar{r}(e) = \nu + \sigma + 1 - \nu\sigma e$. Recalling from (A1) that $e = \frac{1}{\varepsilon}$ and $r = \varepsilon\rho$ gives the result in (13b).

To prove sufficiency, calculate the derivatives of the demand function in (13a):

$$\text{(B10)}\quad p'(x) = -\eta\alpha x^{-\eta-1} - \theta\beta x^{-\theta-1} \quad \text{and} \quad p''(x) = \eta\alpha x^{-\eta-2} + \theta(\theta+1)\beta x^{-\theta-2}$$

Hence the elasticity and convexity can be written as:

$$\text{(B11)}\qquad \varepsilon = \frac{f+1}{\eta f + \theta} \quad \text{and} \quad \rho = \frac{\eta(\eta+1)f + \theta(\theta+1)}{\eta f + \theta}$$

where $f = F(x, \alpha, \beta, \eta, \theta) \equiv \frac{\alpha}{\beta} x^{\eta-\theta}$. Equation (B11) clearly satisfies condition (11a) from Proposition 2 for α and β, so the bipower inverse demand manifold is invariant with respect to these two parameters. Eliminating f gives the explicit expression for the manifold, which completes the proof of sufficiency: $\bar{\rho}(\varepsilon) = \eta + \theta + 1 - \eta\theta\varepsilon$.

To prove necessity, assume the manifold is affine, so $\rho(x) = a + b\varepsilon(x)$ where a and b are constants. Substituting for $\rho(x)$ and $\varepsilon(x)$ and collecting terms yields:

$$\text{(B12)}\qquad x^2 p''(x) + axp'(x) - bp(x) = 0$$

To solve this second-order Euler-Cauchy differential equation, we change variables as follows: $t = \log x$ and $p(x) = g(\log x) = g(t)$. Substituting for $p(x) = g(t)$, $p'(x) = \frac{1}{x} g'(t)$ and $p''(x) = \frac{1}{x^2}\left[g''(t) - g'(t)\right]$ into (B12) gives a linear differential equation:

$$\text{(B13)}\qquad g''(t) + (a-1)g'(t) - bg(t) = 0$$

Assuming a trial solution $g(t) = e^{\lambda t}$ gives the characteristic polynomial: $\lambda^2 + (a-1)\lambda - b = 0$, whose roots are $\lambda = \frac{1}{2}\left[-(a-1) \pm \sqrt{(a-1)^2 + 4b}\right]$. Only real roots make sense, so we assume $(a-1)^2 + 4b \geq 0$. If the inequality is strict, the roots are distinct and the general solution is given by $g(t) = \alpha e^{\lambda_1 t} + \beta e^{\lambda_2 t}$, where α and β are constants of integration. If $(a-1)^2 = -4b$, the roots are equal and the general solution is given by $g(t) = (\alpha + \beta t)e^{\lambda t}$. In both cases, the solution may be found by switching back from t and $g(t)$ to $\log x$ and $p(x)$, recalling that $e^{\lambda \log x} = x^{\lambda}$. Hence, in the first case, $p(x) = \alpha x^{\lambda_1} + \beta x^{\lambda_2}$, and in the second case, $p(x) = (\alpha + \beta \log x)x^{\lambda}$.[4] The final step is to note that the sum of the roots is $\lambda_1 + \lambda_2 = 1 - a$ and their product is $\lambda_1\lambda_2 = b$, which implies the relationship

[4]We do not present the case of equal roots separately in the statement of Proposition 3 in the text: the economic interpretation is more convenient if we view it as the limiting case of the general expression as η approaches zero. See, for example, footnotes 21 and 23, which illustrate this for CARA and translog demands respectively.

between the coefficients of the manifold and those of the implied demand function stated in the proposition. This completes the proof of (13a), while that of (13b) follows immediately by duality, as already noted.

B7. Proof of Proposition 4: Bipower Superconvexity

Substituting from the bipower inverse demand manifold in (13a) into the condition for superconvexity, $\rho \geq \frac{\varepsilon+1}{\varepsilon}$, yields:

$$\rho - \frac{\varepsilon+1}{\varepsilon} = -\frac{1}{\varepsilon}(\eta\varepsilon - 1)(\theta\varepsilon - 1) \tag{B14}$$

Next, following the approach of Proposition 2, we write the elasticity of demand in terms of a sub-function: $\varepsilon = \frac{\alpha x^{-\eta}+\beta x^{-\theta}}{\eta\alpha x^{-\eta}+\theta\beta x^{-\theta}} = \frac{f+1}{\eta f+\theta}$, where $f \equiv \frac{\alpha}{\beta}x^{\theta-\eta}$. Substituting into (B14) yields:

$$\rho - \frac{\varepsilon+1}{\varepsilon} = \left(\frac{\eta-\theta}{\eta f+\theta}\right)^2 \frac{f}{\varepsilon} \tag{B15}$$

Hence, superconvexity requires that f must be positive, and so α and β must have the same sign. Since at least one of them must be positive, this implies that they must both be positive for superconvexity, which proves the first part of Proposition 4.

Similarly, substituting from the bipower direct demand manifold in (13b) into the condition for superconvexity yields:

$$\rho - \frac{\varepsilon+1}{\varepsilon} = -\frac{1}{\varepsilon^2}(\varepsilon - \nu)(\varepsilon - \sigma) \tag{B16}$$

Once again we eliminate the terms in ε in parentheses using $\varepsilon = \frac{\nu\gamma p^{-\nu}+\sigma\delta p^{-\sigma}}{\gamma p^{-\nu}+\delta p^{-\sigma}} = \frac{\nu g+\sigma}{g+1}$, where $g \equiv \frac{\gamma}{\delta}p^{\sigma-\nu}$. This yields:

$$\rho - \frac{\varepsilon+1}{\varepsilon} = \left(\frac{\nu-\sigma}{g+1}\right)^2 \frac{g}{\varepsilon^2} \tag{B17}$$

It follows that both γ and δ must be positive for superconvexity, which proves the second part of Proposition 4.

B8. Examples of Bipower Direct Demands

Properties of Pollak Demands

With $\nu = 0$ in the bipower direct case, the elasticity of demand becomes $\varepsilon = \frac{\sigma\delta p^{-\sigma}}{\gamma+\delta p^{-\sigma}} = \sigma\frac{\sigma\delta p^{-\sigma}}{x} = \sigma\frac{x-\gamma}{x}$. It follows that σ, δ and $x-\gamma$ must have the same sign. The sign of σ also determines whether the inverse demand function is

logconvex or not. The CARA demand function is the limiting case when $\sigma \to 0$: the direct demand function becomes $x = \gamma' + \delta' \log p$, $\delta' < 0$, which implies that the inverse demand function is log-linear: $\log p = \alpha + \beta x$, $\beta < 0$.[5] The CARA manifold is $\bar{\rho}(\varepsilon) = \frac{1}{\varepsilon}$, which is a rectangular hyperbola through the point $\{1.0, 1.0\}$. Hence the CARA function is the dividing line between two sub-groups of demand functions and their corresponding manifolds, with σ either negative or positive. For negative values of σ, γ is an upper bound to consumption: the best-known example of this class is the linear demand function, corresponding to $\sigma = -1$. By contrast, for strictly positive values of σ, γ is the lower bound to consumption and there is no upper bound. Especially in the LES case, it is common to interpret γ as a "subsistence" level of consumption, but this requires that it be positive, which (when σ and δ are positive) only holds if demand is superconvex. All members of the Pollak family with positive σ are translated-CES functions, and, as the arrows in Figure 5(a) indicate, they asymptote towards the corresponding "untranslated-CES" function as sales rise without bound; for example, the LES demand function, with σ equal to one, asymptotes towards the Cobb-Douglas. Table B1 summarizes the three possible cases of this family of demand functions.

Table B1—Properties of Pollak Demand Functions

	$\gamma > 0$	$\gamma < 0$
$\sigma > 0$, $\delta > 0$	1. Superconvex; logconvex: $x > \gamma > 0$	2. Subconvex; logconvex
$\sigma < 0$, $\delta < 0$	3. Subconvex; logconcave: $\gamma > x > 0$	n/a

Pollak showed that these are the only demand functions that are consistent with both additive separability and quasi-homotheticity (so the expenditure function exhibits the "Gorman Polar Form"). Just as (13b) is dual to (13a), so the Pollak family of direct demand functions is dual to the Bulow-Pfleiderer family of inverse demand functions. An implication of this is that, corresponding to the property of Bulow-Pfleiderer demands that marginal revenue is linear in price, Pollak demands exhibit the property that the marginal *loss* in revenue from a small increase in price is linear in sales.[6] This implies that the coefficient of absolute risk aversion for these demands is hyperbolic in sales, which is why, in the theory of choice under uncertainty, they are known as "HARA" ("hyperbolic

[5] As noted by Pollak, this demand function was first proposed by Chipman (1965), who showed that it is implied by an additive exponential utility function. Later independent developments include Bertoletti (2006) and Behrens and Murata (2007). Differentiating the Arrow-Pratt coefficient of absolute risk aversion defined in footnote 7 gives $\frac{\partial A(x)}{\partial x} = -\frac{u'u''' - (u'')^2}{(u'')^2} = -\frac{pp'' - (p')^2}{(p')^2} = 1 - \varepsilon\rho$, so absolute risk aversion is constant if and only if $\varepsilon = \frac{1}{\rho}$.

[6] As we show in Appendix B9 below, Bulow-Pfleiderer demands $p(x) = \alpha + \beta x^{-\theta}$ satisfy the property: $p + xp' = \theta\alpha + (1 - \theta)p$. Switching variables, we can conclude that Pollak demands $x(p) = \gamma + \delta p^{-\sigma}$ satisfy the property: $x + px' = \sigma\gamma + (1 - \sigma)x$.

absolute risk aversion") demands following Merton (1971).[7]

Properties of PIGL Demands

With $\nu = 1$, the elasticity of demand becomes $\varepsilon = \frac{\gamma p^{-1}+\sigma\delta p^{-\sigma}}{\gamma p^{-1}+\delta p^{-\sigma}}$. Subtracting one gives: $\varepsilon - 1 = \frac{(\sigma-1)\delta p^{-\sigma}}{\gamma p^{-1}+\delta p^{-\sigma}} = (\sigma-1)\,\frac{px-\gamma}{px}$. It follows that $\sigma - 1$, δ and $px - \gamma$ must have the same sign. In addition, the demand manifold is $\rho = \frac{(\sigma+2)\varepsilon-\sigma}{\varepsilon^2}$, so convexity is increasing in σ. Combining these results with Proposition 4, there are three possible cases of this demand function. For σ less than one, the demand function is less convex than the translog (i.e., PIGLOG) case, δ is negative and γ is positive. For σ greater than one, δ is positive, the demand function is more convex than the translog case, and it is subconvex if γ is negative, otherwise it is superconvex. These properties are dual to those of the inverse PIGL demand functions in Appendix B9, and, like the latter, they can be related to whether the elasticity of marginal revenue with respect to *price* is greater or less than one (the value of one corresponding to the PIGLOG case). Note finally that the limiting case of PIGL demand function when σ approaches zero is the LES, the only demand function that is a subset of both PIGL and Pollak. The LES case is special in another respect: as can be seen in Figure 5(a), it is the only member of the PIGL family for which ε is monotonic in ρ along the manifold. In all other cases the manifold is vertical at $\{\varepsilon,\ \rho\} = \{\frac{2\sigma}{\sigma+2},\ \frac{(\sigma+2)^2}{4\sigma}\}$. For $\sigma < 0$ it is not defined for $\rho < \frac{(\sigma+2)^2}{4\sigma}$, while for $\sigma > 0$ it is not defined for $\rho > \frac{(\sigma+2)^2}{4\sigma}$.

QMOR Demand Functions

Diewert (1976) introduced the quadratic mean of order r expenditure function, which implies a general functional form for homothetic demand functions. Feenstra (2014) considers a symmetric special case and shows how it can be adapted to allow for entry and exit of goods, so making it applicable to models of monopolistic competition. In our notation, the resulting family of demand functions, taking a "firm's-eye view", is:

$$x\,(p) = \gamma p^{-(1-r)} + \delta p^{-\frac{2-r}{2}} \tag{B18}$$

This is clearly a member of the bipower direct family, with $\nu = 1-r$ and $\sigma = \frac{2-r}{2}$. Hence, from Proposition 3, its demand manifold is:

$$\bar{\rho}\,(\varepsilon) = \frac{(2-r)\,(3\varepsilon-1+r)}{2\varepsilon^2} \tag{B19}$$

[7] The Arrow-Pratt coefficient of absolute risk aversion is $A(x) \equiv -\frac{u''(x)}{u'(x)}$. With additive separability this becomes $A(x) = -\frac{p'(x)}{p(x)} = -\frac{1}{px'(p)}$. Using the result from footnote 6, this implies that with Pollak demands, $A(x) = \frac{1}{\sigma(x-\gamma)}$, which is hyperbolic in x.

In the limit as $r \to 0$, this becomes $\bar{\rho}(\varepsilon) = \frac{3\varepsilon - 1}{\varepsilon^2}$, which is the translog manifold discussed in the text. Figure B2 illustrates this demand manifold for a range of values of r. For $r = 2$ it coincides with the $\rho = 0$ vertical line: i.e., a linear demand function from the firm's perspective. For negative values of r (i.e., more convex than the translog), the manifolds extend into the superconvex region. However, this is for arbitrary values of γ and δ. Feenstra (2014) shows that these parameters, which depend on real income and on prices of other goods, must be of opposite sign when the demand function (B18) is derived from expenditure minimization. Hence, from Proposition 4, QMOR demands are not consistent with superconvexity, though in other respects they allow for considerable flexibility in modeling homothetic demands.

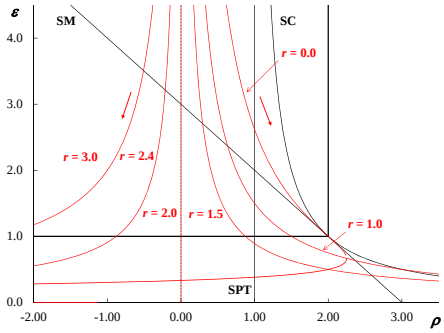


FIGURE B2. DEMAND MANIFOLDS FOR QMOR DEMAND FUNCTIONS

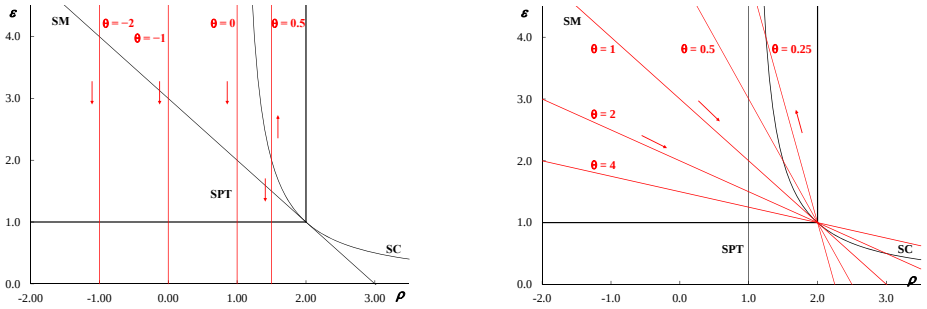
B9. Examples of Bipower Inverse Demands

PROPERTIES OF BULOW-PFLEIDERER DEMANDS

As noted in the text, the first sub-case of the bipower inverse demand functions in (13a) we consider comes from setting η equal to zero, giving the iso-convex or "constant pass-through" family of Bulow and Pfleiderer (1983): $p(x) = \alpha + \beta x^{-\theta}$. Convexity ρ equals a constant $\theta + 1$, so from (7) $\frac{1}{1-\theta}$ measures the degree of absolute pass-through for this system. Pass-through can be more than one-for one, as in the CES case ($\alpha = 0$, $\theta = \frac{1}{\sigma} > 0$); exactly one-for-one, as in the log-linear direct demand case ($\theta \to 0$, so $p(x) = \alpha' + \beta' \log x$, implying that $\log x(p) = \gamma + \delta p$); or less than one-for-one, as in the case of linear demand ($\theta = -1$ so exactly half of a cost increase is passed through to prices).

This family has many other attractive properties. It is necessary and sufficient for marginal revenue to be affine in price. (See below.) It can be given a discrete choice interpretation: it equals the cumulative demand that would be generated

by a population of consumers if their preferences followed a Generalized Pareto Distribution.[8] Finally, as shown by Weyl and Fabinger (2013) and empirically implemented by Atkin and Donaldson (2012), it allows the division of surplus between consumers and producers to be calculated without knowledge of quantities. Figure B3(a) shows the demand manifolds for some members of this family.[9]



(a) The Bulow-Pfleiderer Family (b) The Inverse PIGL Family

FIGURE B3. DEMAND MANIFOLDS FOR SOME BIPOWER INVERSE DEMAND FUNCTIONS

With $\eta = 0$, the elasticity of demand becomes: $\varepsilon = \frac{\alpha+\beta x^{-\theta}}{\theta\beta x^{-\theta}} = \frac{p}{\theta\beta x^{-\theta}} = \frac{p}{\theta(p-\alpha)}$. It follows that θ, β and $p-\alpha$ must have the same sign. The sign of θ also determines whether the direct demand function is logconvex (i.e., whether it exhibits super-pass-through) or not: recall that $\rho-1 = \theta$. There are therefore three possible cases of this demand function: see Table B2. As shown by Bulow and Pfleiderer (1983), these demands are necessary and sufficient for marginal revenue to be affine in price. Sufficiency is immediate: marginal revenue is $p + xp' = \theta\alpha + (1-\theta)p$. Necessity follows by solving the differential equation $p(x) + xp'(x) = a + bp(x)$, which yields $p(x) = \frac{a}{1-b} + c_1 x^{b-1}$, where c_1 is a constant of integration.

TABLE B2—PROPERTIES OF BULOW-PFLEIDERER DEMAND FUNCTIONS

	$\alpha > 0$	$\alpha < 0$
$\theta > 0$, $\beta > 0$	1. Superconvex; logconvex: $p > \alpha > 0$	2. Subconvex; logconvex
$\theta < 0$, $\beta < 0$	3. Subconvex; logconcave: $\alpha > p > 0$	n/a

[8] See Bulow and Klemperer (2012).

[9] Note how the behavior implied by these manifolds differs from the Pollak case in Figure 5(a), especially in the super-pass-through region. With Bulow-Pfleiderer demands, firms diverge from the CES benchmark along the SC locus as sales increase, whereas with Pollak demands they converge towards it; both these statements hold whether demands are super- or subconvex. This allows a simple visualization of the limiting behavior of a monopolistically competitive sector as market size increases without bound.

Properties of Inverse PIGL Demands

The second case of (13a) considered in the text comes from setting η equal to one, which yields the "inverse PIGL" ("price-independent generalized linear") system: $p(x) = \frac{1}{x}(\alpha + \beta x^{1-\theta})$. This system implies that the elasticity of marginal revenue defined in footnote 15 is constant and equal to θ: $\eta = 1$ implies from (13a) that $-\frac{xR''}{R'} = \frac{2-\rho}{\varepsilon-1} = \theta$. The limiting case as $\theta \to 1$ is the inverse "PIGLOG" ("price-independent generalized logarithmic") or inverse translog, $p(x) = \frac{1}{x}(\alpha' + \beta' \log x)$.[10] This implies that the elasticity of marginal revenue is unity, and so, as noted in Mrázová and Neary (forthcoming), it coincides with the supermodularity locus: $\eta = \theta = 1$ implies from (13a) that $\bar{\rho}(\varepsilon) = 3 - \varepsilon$. Figure B3(b) shows the demand manifolds for some members of this family.

With $\eta = 1$, so the elasticity of demand becomes $\varepsilon = \frac{\alpha x^{-1} + \beta x^{-\theta}}{\alpha x^{-1} + \theta \beta x^{-\theta}}$, its value less one can be written in two alternative ways: $\varepsilon - 1 = \frac{(1-\theta)\beta x^{1-\theta}}{\alpha + \theta \beta x^{1-\theta}} = (1-\theta)\frac{px - \alpha}{\theta px + (1-\theta)\alpha}$. It follows that $1 - \theta$, β and $px - \alpha$ must have the same sign. (Recall that θ itself equals $\frac{2-\rho}{\varepsilon-1}$ and so must be positive in the admissible region.) The value of $1 - \theta$ also determines whether the demand function is supermodular or not: substituting from the demand manifold $\bar{\rho}(\varepsilon) = 2 + (1 - \varepsilon)\theta$ into the condition for supermodularity gives $\varepsilon + \rho > 3 \Leftrightarrow (\varepsilon - 1)(1 - \theta) > 0 \Leftrightarrow \theta < 1$. Combining these results with Proposition 4 shows that there are three possible cases of this demand function, as shown in Table B3.

Table B3—Properties of Inverse PIGL Demand Functions

	$\alpha > 0$	$\alpha < 0$
$\theta < 1$, $\beta > 0$	1. Superconvex; supermodular: $px > \alpha > 0$	2. Subconvex; supermodular
$\theta > 1$, $\beta < 0$	3. Subconvex; submodular: $\alpha > px > 0$	n/a

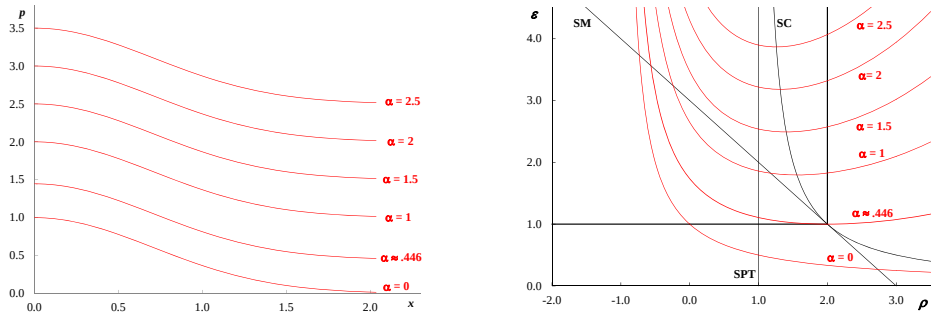
B10. Inverse Exponential Demand

In this section we introduce a demand function, the inverse exponential, which is an example that, for the same parameter values, is sometimes sub- and sometimes superconvex:[11]

$$p(x) = \alpha + \beta \exp(-\gamma x^{\delta}) \tag{B20}$$

[10] To show this, take the limit as in footnote 23.

[11] Mrázová and Neary (forthcoming) consider the properties of R&D cost functions of this form.



(a) Demand Functions (b) Demand Manifolds

FIGURE B4. INVERSE EXPONENTIAL DEMAND FUNCTIONS AND MANIFOLDS

where $\gamma > 0$ and $\delta > 0$. The elasticity and convexity of demand are found to be:

$$\varepsilon(x) = \frac{1 + \frac{\alpha}{\beta}\exp(f)}{\delta f} \quad \text{and} \quad \rho(x) = \delta f - \delta + 1 \tag{B21}$$

where $f \equiv F(x, \alpha) = \gamma x^{\delta}$. Eliminating f yields a closed-form expression for the demand manifold:

$$\bar{\varepsilon}(\rho) = \frac{1 + \frac{\alpha}{\beta}\exp\left(\frac{\rho+\delta-1}{\delta}\right)}{\rho + \delta - 1} \tag{B22}$$

This is invariant with respect to γ, in accordance with Proposition 2, and it also depends only on the ratio of α and β, not on their levels, in accordance with Corollary 2. Differentiating with respect to ρ shows that, provided $\frac{\alpha}{\beta}$ is strictly positive, the demand function is subconvex for low values of ρ, which from (B21) implies low values of x, but superconvex for high ρ and x:

$$\bar{\varepsilon}_{\rho} = \frac{-\delta + \frac{\alpha}{\beta}(\rho - 1)\exp\left(\frac{\rho+\delta-1}{\delta}\right)}{\delta(\rho + \delta - 1)^2} \tag{B23}$$

Figure B4 illustrates some demand functions and the corresponding manifolds from this class for a range of values of α, assuming $\beta = 1$ and $\delta = 2$. A superconvex range in the admissible region is possible only for parameter values such that the minimum point of the manifold lies above the Cobb-Douglas point, $\{\varepsilon, \rho\} = \{1, 2\}$, i.e., only for $\alpha > \beta\delta\exp\left(-\frac{\delta+1}{\delta}\right)$, which for the values of β and δ underlying Figure B4 is approximately $\alpha > 0.446$. For such values of α, the manifolds are horizontal where they cross the SC locus, in accordance with Figure 1(b).

B11. Proof of Lemma 1: Uniqueness of the Translog

We wish to show that the translog is the only demand function with a manifold of the "contiguous bipower" form $\rho = a_1\varepsilon^\kappa + a_2\varepsilon^{\kappa+1}$ that is always both strictly subconvex and strictly supermodular in the interior of the admissible region. The proof proceeds by showing that these conditions require that the demand manifold satisfy three distinct restrictions. These enable us to isolate the translog demand function as the only candidate.

First, it is clear by inspection that, if a demand function is always both subconvex and supermodular, then its manifold must pass through the Cobb-Douglas point, $\{\varepsilon, \rho\} = \{1, 2\}$. Hence the parameters must satisfy $a_1 + a_2 = 2$.

Second, the slope of the manifold, $\frac{d\rho}{d\epsilon} = a_1\kappa\varepsilon^{\kappa-1} + a_2(\kappa+1)\varepsilon^\kappa$, must be greater than that of the SM locus and less than that of the SC locus at $\{1, 2\}$. Both of these slopes equal -1: $\left.\frac{d\rho}{d\epsilon}\right|_{SM} = -1$ everywhere, and $\left.\frac{d\rho}{d\epsilon}\right|_{SC} = -\frac{1}{\varepsilon^2} = -1$ at $\{1, 2\}$. Hence the parameters must satisfy $a_1\kappa + a_2(\kappa+1) = -1$. This and the previous restriction can be solved for a_1 and a_2 in terms of κ: $a_1 = 3 + 2\kappa$ and $a_2 = -(1 + 2\kappa)$.

Third, the curvature of the manifold, $\left.\frac{d^2\rho}{d\epsilon^2}\right|_M = a_1(\kappa-1)\kappa\varepsilon^{\kappa-2} + a_2\kappa(\kappa+1)\varepsilon^{\kappa-1}$, must be greater than that of the SM locus and less than that of the SC locus at $\{1, 2\}$. These curvatures are: $\left.\frac{d^2\rho}{d\epsilon^2}\right|_{SM} = 0$ everywhere, and $\left.\frac{d^2\rho}{d\epsilon^2}\right|_{SC} = \frac{1}{\varepsilon^3} = 1$ at $\{1, 2\}$. Hence the parameters must satisfy $0 \leq a_1(\kappa-1)\kappa + a_2\kappa(\kappa+1) \leq 1$. Substituting for a_1 and a_2 and simplifying gives: $0 \leq -2\kappa(\kappa+2) \leq 1$. Only two integer values of κ satisfy these inequalities: $\kappa = 0$ is the SM locus itself, which is not in the interior of the admissible region; that leaves $\kappa = -2$, implying $\rho = -\varepsilon^{-2} + 3\varepsilon^{-1} = \frac{3\varepsilon-1}{\varepsilon^2}$, the translog demand manifold, as was to be proved.

B12. Demand Functions that are not Manifold-Invariant

In this section we introduce two new demand systems whose demand manifolds can be written in closed form, though they depend on all the parameters, and so are not manifold invariant. We consider in turn: the "Doubly-Translated CES" super-family, which nests both the Pollak and Bulow-Pfleiderer families; and the "Translated Bipower Inverse" super-family, which nests both the "APT" (Adjustable Pass-Through) system of Fabinger and Weyl (2012) and a new family that we call the inverse "iso-temperance" system.[12]

[12] A third super-family is the dual of the second, the "Translated Bipower Direct" super-family. Reversing the roles of p and x in equation (B26) below leads to a "dual" manifold giving the inverse elasticity e as a function of the direct convexity r with the same form as (B28). Special cases of this include the dual of the APT system and the direct "iso-temperance" system (i.e., the demand system necessary and sufficient for $-px'''/x''$ to be constant). It does not seem possible to express the manifold $\bar{\varepsilon}(\rho)$ in closed form for this family.

The "Doubly-Translated CES" Super-Family

We can nest the Pollak and Bulow-Pfleiderer families as follows: $p(x) = \alpha + \beta (x-\gamma)^{-\theta}$.[13] The elasticity and convexity of this function are:

$$\varepsilon(x) = \frac{1}{\theta}\frac{p}{p-\alpha}\frac{x-\gamma}{x} \qquad \rho(x) = (\theta+1)\frac{x}{x-\gamma} \tag{B24}$$

When γ is zero this reduces to the Bulow-Pfleiderer case. Assuming $\gamma \neq 0$, we have $\rho \neq \theta + 1$, and so the expression for ρ in (B24) can be solved for x: $x = \frac{\rho}{\rho-(\theta+1)}\gamma$. Substituting into the expression for ε yields:[14]

$$\bar{\varepsilon}(\rho) = \left[1 + a_1\left(\frac{1}{\rho - a_2}\right)^{a_3}\right]\frac{a_4}{\rho} \tag{B25}$$

where $a_1 = \frac{\alpha}{\beta}\{(\theta+1)\gamma\}^{\theta}$, $a_2 = \theta+1$, $a_3 = \theta$, and $a_4 = \frac{\theta+1}{\theta}$. This is a closed-form expression for the manifold but it depends on all four parameters, except in special cases such as the Pollak family, when, with $\alpha = 0$, it reduces to $\bar{\varepsilon}(\rho) = \frac{\theta+1}{\theta}\frac{1}{\rho}$. Nevertheless, the general demand manifold (B25) allows for considerable economy of information: three of its four parameters depend only on the exponent θ in the demand function, and the fourth parameter, a_1, is invariant to rescalings of the demand function parameters which keep $\frac{\alpha}{\beta}\gamma^{\theta}$ constant.

The "Translated Bipower Inverse" Super-Family

This demand function adds an intercept α_0 to the bipower inverse family given by (13a):

$$p(x) = \alpha_0 + \alpha x^{-\eta} + \beta x^{-\theta} \tag{B26}$$

Differentiating gives the elasticity and convexity:

$$\varepsilon(x) = \frac{\alpha_0 x^{\eta} + \alpha + \beta x^{\eta-\theta}}{\eta\alpha + \theta\beta x^{\eta-\theta}} \qquad \rho(x) = \frac{\eta(\eta+1)\alpha + \theta(\theta+1)\beta x^{\eta-\theta}}{\eta\alpha + \theta\beta x^{\eta-\theta}} \tag{B27}$$

Assuming as before that $\rho \neq \theta + 1$, and also that $\eta \neq \theta$, we can invert $\rho(x)$ to solve for x: $x(\rho) = \left[\frac{\eta\alpha}{\theta\beta}\frac{(\eta+1)-\rho}{\rho-(\theta+1)}\right]^{\frac{1}{\eta-\theta}}$. Substituting into $\varepsilon(x)$ gives a closed-form expression for the manifold:

$$\bar{\varepsilon}(\rho) = \frac{\rho - a_1}{a_2} + (a_3 - \rho)^{a_4}(\rho - a_5)^{a_6}a_7 \tag{B28}$$

[13] After we developed this family, we realized that it had already been considered in the working paper version of Zhelobodko et al. (2012), who call it the "Augmented-HARA" system.

[14] Here and elsewhere, the parameters must be such that, when the exponent (here θ) is not an integer, the expression which is raised to the power of that exponent is positive.

where $a_1 = \eta + \theta + 1$, $a_2 = -\eta\theta$, $a_3 = \eta + 1$, $a_4 = \frac{\eta}{\eta-\theta}$, $a_5 = \theta + 1$, $a_6 = -\frac{\theta}{\eta-\theta}$, and $a_7 = \left(\frac{\eta}{\beta}\right)^{\frac{\eta}{\eta-\theta}}\left(\frac{\theta}{\alpha}\right)^{-\frac{\theta}{\eta-\theta}}\frac{\alpha_0}{\eta\theta(\eta-\theta)}$. In general, this depends on the same five parameters as the demand function (B26), though once again it allows for considerable economy of information: all but a_7 depend only on the two exponents η and θ, and a_7 itself is unaffected by changes in the other three demand-function parameters that keep $\alpha^{\frac{\theta}{\eta-\theta}}\beta^{\frac{-\eta}{\eta-\theta}}\alpha_0$ constant. Equation (B28) is best understood by considering some special cases:

(1) Bipower Inverse: The cost in additional complexity of the "translation" parameter α_0 is apparent. Setting this equal to zero, the expression simplifies to give the bipower inverse manifold as in Proposition 3: $\bar{\rho}(\varepsilon) = 1 + \eta + \theta - \eta\theta\varepsilon$.

(2) APT Demands: Fabinger and Weyl (2012) show that the pass-through rate (in our notation, $\frac{dp}{dc} = \frac{1}{2-\rho}$) is quadratic in the square root of price if and only if the inverse demand function has the form of (B26) with $\eta = 2\theta$. This reduces the number of parameters by one, so the demand manifold simplifies to: $\bar{\varepsilon}(\rho) = \frac{1+3\theta-\rho}{2\theta^2} - \frac{[(2\theta+1)-\rho]^2}{\rho-(\theta+1)}\frac{2\alpha}{\beta^2\theta^2}\alpha_0$.

(3) Iso-Temperance Demands: Setting $\eta = -1$ is sufficient to ensure that temperance, $\chi \equiv -\frac{xp'''}{p''}$, is constant, equal to $\theta + 2$. It is also necessary. To see this, write $xp''' = -\chi p''$, where χ is a constant, and integrate three times, which yields $p(x) = c_0 + c_1 x + \frac{c_2}{(1-\chi)(2-\chi)}x^{2-\chi}$, where c_0, c_1 and c_2 are constants of integration. This is identical to (B26) with $\eta = -1$ and $\theta = \chi - 2$. Note that iso-convexity implies iso-temperance, but the converse does not hold; just as CES implies iso-convexity, but the converse does not hold.

These special cases and the general demand manifold in (B28) allow us to infer the comparative statics implications of this family of demand functions. Moreover, if we are mainly interested in pass-through, we do not need to work with the demand manifold at all, since the key conditions in (7) and (B4) do not depend on the elasticity of demand (a point stressed by Weyl and Fabinger (2013)). In such cases, our approach can be applied to the *slope* rather than the *level* of demand. By relating the elasticity and convexity of this slope to each other, we can construct a "demand-slope manifold" corresponding to any given demand function, and the properties of this manifold are very informative about when pass-through is increasing or decreasing with sales. In ongoing work, we show that the demand-slope manifolds of the APT and iso-temperance demand functions are particularly convenient in this respect.

B13. Calculating the Effects of Globalization

To solve for the results in (24), use (22) to eliminate $\hat{x}$ from (20) and then solve (20) and (21) for $\hat{p}$ and $\hat{y}$, with $\hat{n}$ determined residually by (23). The results in (25) are obtained by using $\hat{x} = \hat{y} - \hat{k}$ and $\hat{N} = \hat{k} + \hat{n}$.

These results are for infinitesimal changes only. For finite changes, it is still true that the values of ε and ρ determine the results. However, their values are

not fixed in general, so it is necessary to integrate the change in the dependent variable along a path taking account of the changes in ε and ρ.

B14. Change in Real Income: Details

With symmetric preferences and identical prices for all goods, the budget constraint becomes: $I = \int_0^N p(\omega)x(\omega)\,\mathrm{d}\omega = Npx$. So consumption of each good is: $x = \frac{I}{Np}$. Substituting into the direct utility function yields its indirect counterpart:

$$V(N,p,I) = F\left[Nu\left(\frac{I}{Np}\right)\right] \tag{B29}$$

We can now define equivalent income $Y(N,p)$ as the income that preserves the initial level of utility U_0 following a shock:

$$V\left(N,p,\frac{I}{Y}\right) = U_0 \tag{B30}$$

For small changes (so equivalent and compensating variations coincide), we logarithmically differentiate, with I fixed (since it equals exogenous labor income), to obtain: $\hat{N} - \xi(\hat{N} + \hat{p} + \hat{Y}) = 0$. Rearranging gives the change in real income in (33). Note that this is independent of the function F.

B15. Welfare with Bulow-Pfleiderer Preferences

The Bulow-Pfleiderer sub-utility function in (35) takes a bipower form. Hence we can immediately apply equation (13a) from Proposition 3 in Section II.D, replacing η by -1, θ by $\theta - 1$, ρ by $\frac{1}{\varepsilon}$ and ε by $-\frac{1}{\xi}$:

$$u(x) = \alpha x + \frac{1}{1-\theta}\beta x^{1-\theta} \quad \Leftrightarrow \quad \frac{1}{\varepsilon} = \theta + (\theta - 1)\left(-\frac{1}{\xi}\right) \tag{B31}$$

Rearranging gives the first equation in (36), and using the demand manifold to eliminate θ gives the second.

B16. Welfare with Pollak Preferences

From Demands to Preferences

Recall from Section II.D that the Pollak demand function is $x(p) = \gamma + \delta p^{-\sigma}$, where δ, σ and $x - \gamma$ have the same sign, and γ has the same sign as δ and σ if and only if demand is superconvex. To derive the sub-utility function we must first invert to obtain the inverse demand function. This yields: $p(x) = \left(\frac{x-\gamma}{\delta}\right)^{-\frac{1}{\sigma}}$. It is convenient to redefine the constants as $\zeta \equiv -\gamma\sigma$ and $\beta \equiv (\delta/\sigma)^{1/\sigma}$ (i.e., we

replace γ by $-\zeta/\sigma$, and δ by β^δ/σ), which yields: $p(x) = \beta(\sigma x + \zeta)^{-\frac{1}{\sigma}}$. Both β and $\sigma x + \zeta$ are positive. Integrating and setting the constant of integration equal to zero yields the sub-utility function (37). We have already seen in Proposition 4 that all bipower direct demand functions are superconvex if and only if both γ and δ are positive, and in Section B8 that with Pollak demands σ and δ must have the same sign. Hence a negative value of $\zeta \equiv -\gamma\sigma$ is necessary and sufficient for demand to be superconvex.

Gains and Losses from Globalization with Pollak Preferences

Substituting for $\xi = \frac{\varepsilon\rho - 2}{\varepsilon}$ into the general expression for welfare change in equation (34) gives in this case:

$$\hat{Y} = \frac{\varepsilon}{\varepsilon\rho - 2}\left[1 - \frac{(\varepsilon - 1)^2}{\varepsilon^2(2 - \rho)}\right]\hat{k} \tag{B32}$$

This is negative when $\rho > \rho^Y \equiv \frac{\varepsilon^2 + 2\varepsilon - 1}{\varepsilon^2}$. To confirm that this lies in the admissible range $\rho \in \left[\underline{\rho}, \overline{\rho}\right] \equiv \left[\frac{2}{\varepsilon}, 1 + \frac{2}{\varepsilon}\right]$, note that $\rho^Y - \underline{\rho} = \frac{\varepsilon^2 - 1}{\varepsilon^2} > 0$ and $\overline{\rho} - \rho^Y = \frac{1}{\varepsilon^2} > 0$.

Alternative Normalizations of the Sub-Utility Function

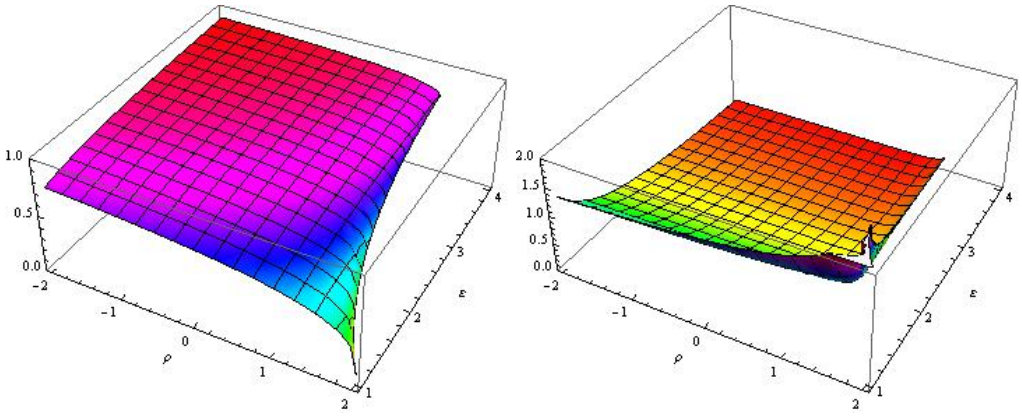
In the text we follow Pollak (1971) and Dixit and Stiglitz (1977) and set the constant of integration in the sub-utility function equal to zero. As Dixit and Stiglitz (1979) point out, this need not imply that $u(0)$ is strictly positive: we can define $u(x) = \max\{0, \frac{\beta}{\sigma - 1}(\sigma x + \zeta)^{\frac{\sigma - 1}{\sigma}}\}$, which is discontinuous at $x = 0$, but in all respects is a valid utility index. However, different authors take different views on whether it is satisfactory that new goods provide a finite level of utility, even when they are consumed in infinitesimal (though strictly positive) amounts. An alternative approach, due to Pettengill (1979), is to choose the constant of integration itself to ensure that $u(0) = 0$. This implies that the sub-utility function takes the following form:

$$u(x) = \frac{\beta}{\sigma - 1}\left[(\sigma x + \zeta)^{\frac{\sigma - 1}{\sigma}} - \zeta^{\frac{\sigma - 1}{\sigma}}\right] \tag{B33}$$

Note, however, that ζ must be positive, which implies as already noted that demand is always subconvex: a zero level of consumption is not in the consumer's feasible set if ζ is negative. Hence this normalization of the Pollak utility function implies a different restriction on the feasible region from that in the text, with the whole of the superconvex region now inadmissible.

The elasticity and convexity of demand are unaffected by this re-normalization of the sub-utility function. However, the elasticity of utility is very different.[15] It

[15]Utility is ordinal, so preferences and demands are invariant to monotonic transformations of the



(a) Elasticity of Utility (b) Change in Real Income

FIGURE B5. GLOBALIZATION AND WELFARE: NORMALIZED POLLAK PREFERENCES

now behaves more like the Bulow-Pfleiderer case, except that it is not consistent with superconvex demands:

$$\text{(B34)} \qquad \xi^N = H\xi, \quad H(\varepsilon,\sigma) = \frac{1}{1-\left(\frac{\varepsilon-\sigma}{\varepsilon}\right)^{\frac{\sigma-1}{\sigma}}}, \quad H(\varepsilon,\rho) = \frac{1}{1-\left(\frac{\varepsilon-\varepsilon\rho+1}{\varepsilon}\right)^{\frac{\varepsilon\rho-2}{\varepsilon\rho-1}}}$$

where H is a correction factor applied to the unnormalized elasticity of utility given in equation (38). The results are shown in Figure B5. Compared with Figure 9 in the text, the main differences are that the elasticity of utility now lies between zero and one for all admissible values of ε and ρ, i.e., throughout the subconvex region, and that the gains from globalization are always positive. Both the elasticity of utility and the change in real income behave qualitatively with respect to ε and ρ in a similar fashion to the case of Bulow-Pfleiderer preferences in Figure 8. All this confirms that the elasticity and convexity of demand are not sufficient statistics for the welfare effects of globalization, and that small changes in the parameterization of utility can have major implications for the quantitative effects of changes in the size of the world economy.

B17. Markup and Pass-Through Data

Table B4 summarizes the data on the markups m and pass-through elasticities k from De Loecker et al. (2016) that we use and gives the implied values of ε and ρ. The mean and median estimates of the markup m are taken from their

overall utility function, i.e., to different choices of the F function, in equation (15). However, utility and its derivatives, and hence the measured gains from trade, are not invariant to monotonic transformations of the sub-utility function, as here.

Table VI (p. 483). They measure markups as $\frac{p}{c}$, which we here adjust to $m \equiv \frac{p-c}{c}$. The estimated pass-through coefficients k are taken from their Table VII (p. 488). From Column (1) of that table we take the OLS estimate of 0.337 with a standard error of 0.041, implying a 95% confidence interval of 0.257 to 0.417. From Column (2), which instruments marginal costs with input tariffs and lagged marginal costs, we take the IV estimate of 0.305 with a standard error of 0.084, implying a 95% confidence interval of 0.140 to 0.470. Column (3) instruments marginal costs with input tariffs and two-period lagged marginal costs: it yields a point estimate that lies within the OLS confidence interval but is much less precisely estimated, with a 95% confidence interval that implies values for ρ extending outside the admissible region.

The OLS estimates are biased when marginal costs and prices are jointly determined. However, they may nevertheless be of interest for two reasons. First, while the results from de Loecker et al. (2016) showing the effects of output tariffs on markups use a second-order polynomial to control for marginal costs, they note that the results are very similar if marginal costs are assumed to be constant. (See their page 494, footnote 53.) Second, while the OLS estimate of the pass-through elasticity may be biased in principle, it is not very different from the IV estimate in practice: recall that the point estimates are 0.337 and 0.305 respectively, and each is comfortably within the confidence interval of the other.

Table B4—Values of ε and ρ implied by Data on m and k

	OLS Estimate of k		IV Estimate of k	
	Mean m	Median m	Mean m	Median m
m	1.70	0.34	1.70	0.34
k^*	0.337 (0.257, 0.417)	0.337 (0.257, 0.417)	0.305 (0.140, 0.470)	0.305 (0.140, 0.470)
ε	1.588	3.941	1.588	3.941
ρ^*	0.901 (0.557, 1.113)	-0.214 (-0.908, 0.212)	0.786 (-0.639, 1.211)	-0.447 (-3.317, 0.411)

Note: Sources: m and k: De Loecker et al. (2016); ε and ρ: authors' calculations (see text). * Point estimates, with implied 95% confidence intervals in parentheses.

B18. Pigou's Law

Following Deaton (1974), assume an additively separable utility function with a finite number of goods:

$$U(x) = F\left[\Sigma_i\, u(x_i)\right] \tag{B35}$$

Maximizing this subject to the budget constraint yields the first-order conditions:

$$u'(x_i) = \lambda p_i \tag{B36}$$

where λ is the marginal utility of income adjusted for the units of measurement of utility in (B35) by deflating by F'. Totally differentiating this and expressing the results in terms of proportional changes gives:

$$\varepsilon_{u'x_i}\hat{x}_i = \hat{\lambda} + \hat{p}_i \tag{B37}$$

where $\varepsilon_{u'x_i} \equiv \frac{d\log u'(x_i)}{d\log x_i}$ is the elasticity of the slope of the sub-utility function with respect to the consumption of good x_i.

If the exogenous shock to the consumer is a change in income (so $\hat{p}_i = 0$), equation (B37) implies:

$$\varepsilon_{u'x_i}\eta_i = \Phi^{-1} \tag{B38}$$

where η_i is the income elasticity of demand for good i and $\Phi \equiv \left[\frac{d\log\lambda}{d\log I}\right]^{-1}$ is the inverse of the elasticity of the marginal utility of income with respect to income, or Frisch's "flexibility of the marginal utility of money". If instead the shock is a change in the price of good j, equation (B37) implies:

$$\varepsilon_{u'x_i}\varepsilon_{ij} = \varepsilon_{\lambda j} + \delta_{ij} \tag{B39}$$

where ε_{ij} is the cross-price elasticity of demand for good i with respect to the price of good j, and δ_{ij} is the Kronecker delta. Multiplying (B39) by the budget share of good i, ω_i, and summing over all goods j yields:

$$-\varepsilon_{u'x_i}\omega_j = \varepsilon_{\lambda j} + \omega_j\eta_j \tag{B40}$$

where we use the aggregation conditions $\Sigma_i\omega_i = 1$ and $\Sigma_i\omega_i\varepsilon_{ij} = -\omega_j$. Finally, substituting into (B38) and eliminating $\varepsilon_{u'x_i}$ yields:

$$\varepsilon_{ij} = \left[\delta_{ij} - \omega_j(\Phi^{-1} + \eta_j)\right]\Phi\eta_i \tag{B41}$$

When $i = j$, this gives the desired relationship between the own-price and income elasticities. It is approximately proportional, with the deviation from proportionality depending on the budget share of good i. In the continuum case, the budget share of any good is infinitesimal and so (B41) with $i = j$ reduces to exact proportionality as in (41) in the text. Note also that, in the continuum case, $\varepsilon_{u'x_i}$ is the inverse of the demand elasticity ε_{ii}: when the budget share is infinitesimal, the Frisch marginal-utility-compensated elasticity of demand, like the Hicksian utility-compensated elasticity, is identical to the Marshallian elasticity of demand.

B19. Glossary of Terms

In this appendix we note some alternative definitions of terms that we use in the text. The text can be read independently of the glossary.

Log-Convexity: We describe a function $f(x)$ as log-convex at a point $(x_0, f(x_0))$ if and only if $\log f(x)$ is convex in x at $(x_0, f(x_0))$. This appears to be standard practice, though there are notable exceptions: see "Superconvexity" below.

Manifold: Each of the demand manifolds we present is a one-dimensional smooth manifold, or a smooth plane curve in the Euclidean plane R^2. Each is defined by an equation $f(\varepsilon, \rho) = 0$, where $f : R^2 \to R$ is a smooth function, and the partial derivatives $\frac{\partial f}{\partial \varepsilon}$ and $\frac{\partial f}{\partial \rho}$ are never both zero. Strictly speaking, a manifold cannot have a self-intersection point, whereas the relationship between ε and ρ could exhibit such a feature.

Marshall's Second Law: In Book III, Chapter IV of his *Principles*, entitled "The Law of Elasticity," Marshall argued that the elasticity of demand increases with price. This is equivalent to what we call subconvexity, and is sometimes called "Marshall's Second Law of Demand." His First Law is, of course, that demand curves slope downwards. (A nice irony is that violations of both laws are of economic interest.) Note that this is different from Marshall's second law of *derived* demand: the demand for an input is likely to be less elastic the smaller its share in the cost of the output which uses it.

Pollak or HARA Demands: The demand functions due to Pollak (1971) that we consider in Section II.D are sometimes called "HARA" ("hyperbolic absolute risk aversion") demands following Merton (1971). In the present context the former label seems more appropriate. Pollak characterized the preferences that are consistent with these demands in a non-stochastic multi-good setting, showing that they are the only ones that are consistent with both additive separability and quasi-homotheticity. By contrast, Merton focused on portfolio allocation in a stochastic one-good setting.

Superconvexity: Following Mrázová and Neary (forthcoming), we describe a function $f(x)$ as superconvex at a point $(x_0, f(x_0))$ if and only if $\log f(x)$ is convex in $\log x$ at $(x_0, f(x_0))$. Arkolakis et al. (forthcoming) use the term "log-convex" for such a function, whereas Kingman (1961) uses the term "superconvex" as a synonym for the more widely-used sense of log-convexity, i.e., $\log f(x)$ convex in x.

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